Unexpected algebraic and geometric properties of Fermat-type configurations

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Motivation

Problem (Containment problem)

Determine positive integers m, r such that

$$I^{(m)} \subset I^r \tag{1}$$

for all homogenous ideals $I \subset \mathbb{K}[x_0, \ldots, x_N]$.

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Problem

To what extend can the bound Nr in the Theorem be lowered?

Fermat arrangement of lines

Definition

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Theorem (Dumnicki-Szemberg-Tutaj-Gasińska, Seceleanu)

For $n \geqslant 3$ the ideal of intersection points of lines in the Fermat arrangement provides a non-containment example for

$$I^{(3)} \subset I^2.$$

Containment and codimension

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$$F_{N,n} = \prod_{0 \leqslant i < j \leqslant N} (x_i^n - x_j^n).$$

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Theorem (Malara-Szpond)

Let I be the ideal of codimension 2 flats determined by intersecting at least 3 hyperplanes in $\mathcal{F}_N^n$ with $N \geqslant 2$ and $n \geqslant 3$. Then the containment

$$I^{(3)} \subset I^2 \quad \textit{fails.}$$

Unexpected hypersurfaces

Definition

We say that a reduced set of points $Z \subset \mathbb{P}^N$ *admits an unexpected hypersurface* of degree d if there exists a sequence of integers $m_1, \dots, m_s$ such that for general points $P_1, \dots, P_s$ the zero-dimensional subscheme $m_1P_1 + \dots + m_sP_s$ fails to impose independent conditions on forms of degree d vanishing along Z.

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Theorem (Cook II, Harbourne, Migliore, Nagel)

If $n \geqslant 5$, the dual set of points of $\mathcal{F}_2^n$ admits an unexpected curve of degree $n+2$ with $m_1 = n+1$. Moreover, this curve is unique and irreducible.

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Theorem (Sz., Szpond)

If $n \geqslant 5$, the set of points determined by $\mathcal{F}_2^n$ admits an unexpected curve of degree $n+2$ with $m_1 = 4$. Moreover, this curve is unique and irreducible.

Fermat-type configuration of points

Definition

A Fermat-type configuration W^n_N of points in $\mathbb{P}^N$ of degree n consists of

- complete intersection points determined by

$$(x_0^n - x_1^n,\ \dots,\ x_{N-1}^n - x_N^n);$$

- all coordinate points.

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Lemma

The ideal of W_N^n is generated by all forms of the type

$$x_i(x_j^n - x_k^n),$$

where $i, j, k \in \{0, \ldots, N\}$ are mutually distinct.

Fermat-type configuration of points in $\mathbb{P}^3$

We study the ideal I generated by the following 8 binomials of degree 4:

$$x(y^3 - z^3),\ x(z^3 - w^3),\ y(x^3 - z^3),\ y(z^3 - w^3),$$

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This is the ideal of $27 = 3^3$ (complete intersection) points of the form

$$P_{(\alpha,\beta,\gamma)} = (1 : \varepsilon^\alpha : \varepsilon^\beta : \varepsilon^\gamma)$$

where ε is a primitive root of unity of order 3 and $1 \leqslant \alpha, \beta, \gamma \leqslant 3$; and the 4 coordinate points. We denote the set of all these 31 points by W_3^3.

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This is a proper subset of points determined by the arrangement $\mathcal{F}_3^3$.

Unexpected surface in $\mathbb{P}^3$

Theorem (Bauer, Malara, Sz., Szpond)

Let $P = (a : b : c : d)$ be a generic point in $\mathbb{P}^3$. Then the quartic

$$\begin{aligned} Q_R(x:y:z:w) = {} & b^2(c^3-d^3)\cdot x^3y + a^2(d^3-c^3)\cdot xy^3 \\ & + c^2(d^3-b^3)\cdot x^3z + c^2(a^3-d^3)\cdot y^3z \\ & + a^2(b^3-d^3)\cdot xz^3 + b^2(d^3-a^3)\cdot yz^3 \\ & + d^2(b^3-c^3)\cdot x^3w + d^2(c^3-a^3)\cdot y^3w \\ & + d^2(a^3-b^3)\cdot z^3w + a^2(c^3-b^3)\cdot xw^3 \\ & + b^2(a^3-c^3)\cdot yw^3 + c^2(b^3-a^3)\cdot zw^3 \end{aligned}$$

- *vanishes at all points of W_3^3,*
- *vanishes to order 3 at P,*
- *is an unexpected surface for W_3^3 with $m_1 = 3$.*

Higher dimensional projective spaces and more fat points

Theorem (Szpond, on arXiv on Tuesday)

Let $N = 2k + 1$ be an odd number. Let W_N^3 be the Fermat-type configuration of points.
Let R and $P_1, \ldots, P_{k-1}$ be generic points in $\mathbb{P}^N$. Then there exists a **unique** *quartic hypersurface*

- *vanishing at all points of W_N^3,*
- *vanishing to order 3 at R,*
- *vanishing to order 2 at $P_1, \ldots, P_{k-1}$,*
- *unexpected for W_N^3 with $m_1 = 3$ and $m_2 = \ldots = m_{k-1} = 2$.*

Example: 2 fat points in $\mathbb{P}^5$

Theorem (Szpond)

Let $P = (b_0 : b_1 : \ldots : b_5)$ be a general point in $\mathbb{P}^5$. Then there exists a unique quartic $Q_{R,P}$ vanishing at

- *all Fermat points W_5^3,*
- *point $R = (a_0 : a_1 : \ldots : a_5)$ to order 3,*
- *point P to order 2.*

Example continued

$$Q_{R,P}(\mathbf{a},\mathbf{b},\mathbf{x}) = \sum_{i=0}^{5} \sum_{j=i+2}^{i+5} h_{i,j}(\mathbf{a},\mathbf{b}) \cdot x_i (x_{i+1}^3 - x_j^3).$$

Example continued

$$Q_{R,P}(\mathbf{a},\mathbf{b},\mathbf{x})=\sum_{i=0}^{5}\sum_{j=i+2}^{i+5} h_{i,j}(\mathbf{a},\mathbf{b})\cdot x_i(x_{i+1}^3-x_j^3).$$

$$h_{i,j}=(-1)^j\cdot Q_R(b_{\alpha_{i,j}[1]},b_{\alpha_{i,j}[2]},b_{\alpha_{i,j}[3]},b_{\alpha_{i,j}[4]})a_i^2,$$

where

- Q_R *is the unexpected quartic in* $\mathbb{P}^3$*;*
- $\alpha_{i,j}=\sigma_j(i+1,i+2,i+3,i+4,i+5)$*;*
- σ_j *removes the entry "j" from a sequence.*

Asymptotic invariants

Theorem (Dumnicki-Harbourne-Nagel-Seceleanu-Szemberg-Tutaj-Gasińska)

For the Fermat configuration W_2^n we have

- *the resurgence* $\rho(I_{2,n}) = \frac{3}{2}$;
- *the asymptotic resurgence* $\widehat{\rho}(I_{2,n}) = \frac{n+1}{n}$;
- *the Waldschmidt constant* $\widehat{\alpha}(I_{2,n}) = n$.

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Work in progress: Farnik, Guardo, Malara, Szpond, Tutaj-Gasińska

For the Fermat configuration W_N^n with $N \geqslant 3$ we have

- *the resurgence* $\rho(I_{N,n}) = \frac{4}{3}$;
- *the asymptotic resurgence* $\widehat{\rho}(I_{2,n}) =?$;
- *the Waldschmidt constant* $\widehat{\alpha}(I_{2,n}) = n$.


thank
you!