Local effectivity in projective spaces

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Schneider-Lang Theorem in one variable

Theorem

Let $f_1, \ldots, f_k$ be meromorphic functions in $\mathbb{C}$ with f_1, f_2 algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $j = 1, \ldots, k$

$$f_j' \in \mathbb{K}[f_1, \ldots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C} \,:\, z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \ldots, k\}$$

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Corollary (Hermite-Lindemann)

For $\omega \in \mathbb{C}^*$ at least one of the numbers ω, $\exp(\omega)$ is transcendental.

Schwarz Lemma in one variable

Theorem

Let f be an analytic function in a disc $\{|z| \leqslant R\} \subset \mathbb{C}$ with at least N zeroes in a disc $\{|z| \leqslant r\}$ with $r < R$. Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^N |f|_R,$$

where

$$|f|_\gamma = \sup_{|z| \leqslant \gamma} |f(z)|.$$

Theorem (Bombieri 1970)

Let $f_1, \ldots, f_k$ be meromorphic functions in $\mathbb{C}^n$ with $f_1, \ldots, f_{n+1}$ algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $i = 1, \ldots, n$, $j = 1, \ldots, k$

$$\frac{\partial}{\partial z_i} f_j \in \mathbb{K}[f_1, \ldots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C}^n \ : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \ldots, k\}$$

is contained in an algebraic hypersurface.

Hörmander version of Schwarz lemma in several variables

Theorem

Let $S \subset \mathbb{C}^n$ be a finite set. Let m be a positive integer. There exists $M(m) > 0$ such that there exists $r > 0$ such that for $R > r$ and a function f analytic in the ball $\{|z| \leqslant R\} \subset \mathbb{C}^n$ vanishing with multiplicity $\geqslant m$ at each point of S

$$|f|_r \leqslant \left(\frac{c(n) \cdot r}{R}\right)^{M(m)} |f|_R,$$

where $c(n)$ is a constant depending only on n.

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Make the statement effective. In particular: what is the maximal value of $M(m)$?

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Theorem (Moreau)

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$$|f|_r \leqslant \left(\frac{\exp(n) \cdot r}{R}\right)^{\alpha(mS)} |f|_R,$$

where $\alpha(mS)$ is the initial degree of $I_S^{(m)}$.

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where $\alpha(mS)$ is the initial degree of $I_S^{(m)}$.

Remark

The constant $\alpha(mS)$ is optimal.

Waldschmidt decomposition

Definition (Waldschmidt decomposition in $\mathbb{P}^N$)

Let $H \cong \mathbb{P}^{N-1}$ be a hyperplane in $\mathbb{P}^N$ and let Z be a subscheme in H. Let D be a divisor of degree d in $\mathbb{P}^N$. The *Waldschmidt decomposition of D with respect to H and Z* is the sum of $\mathbb{R}$-divisors

$$D = D' + \lambda \cdot H$$

such that $\deg(D') = d - \lambda$,

$$\frac{d-\lambda}{\operatorname{mult}_Z D'} \geqslant \widehat{\alpha}(H; \mathcal{O}_H(1), Z) \tag{1}$$

and λ is the least non-negative real number such that (1) is satisfied.

Lower bound on Waldschmidt constants

Theorem (Dumnicki, Sz., Szpond)

Let $H_1, \ldots, H_s$ be $s \geqslant 2$ mutually distinct hyperplanes in $\mathbb{P}^N$. Let $a_1 \geqslant \ldots \geqslant a_s > 1$ be real numbers such that

$$\left\{\begin{array}{c} a_1 - 1 > 0 \\ a_1 a_2 - a_1 - a_2 > 0 \\ \vdots \\ a_1 \ldots a_{s-1} - \sum_{i=1}^{s-1} a_1 \ldots \widehat{a_i} \ldots a_{s-1} > 0 \end{array}\right.$$

and

$$a_1 \ldots a_s - \sum_{i=1}^{s} a_1 \ldots \widehat{a_i} \ldots a_s \leqslant 0.$$

Lower bound on Waldschmidt constants

Theorem (Dumnicki, Sz., Szpond)

Let

$$Z_i = \{P_{i,1}, \ldots, P_{i,r_i}\} \in H_i \setminus \bigcup_{j \neq i} H_j$$

be the set of r_i points such that

$$\widehat{\alpha}(H_i; Z_i) \geqslant a_i$$

and let $Z = \bigcup_{i=1}^{s} Z_i$. Finally, let

$$q := \frac{a_1 \ldots a_{s-1} - \sum_{i=1}^{s-1} a_1 \ldots \widehat{a_i} \ldots a_{s-1}}{a_1 \ldots a_{s-1}} \cdot a_s + s - 1.$$

Then

$$\widehat{\alpha}(\mathbb{P}^N; Z) \geqslant q.$$

Proof ingredients

We assume to the contrary that there is a divisor D of degree d in $\mathbb{P}^N$ vanishing to order at least m at all points of Z such that

$$p := \frac{d}{m} < q.$$

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Let $\Gamma = \Gamma' + \sum_{i=1}^{s}\lambda_i H_i$ be the Waldschmidt decomposition of Γ.

$$\begin{cases} p - \sum\limits_{i=1}^{s}\lambda_i & \geqslant \quad a_1(1-\lambda_1) \\ p - \sum\limits_{i=1}^{s}\lambda_i & \geqslant \quad a_2(1-\lambda_2) \\ & \vdots \\ p - \sum\limits_{i=1}^{s}\lambda_i & \geqslant \quad a_s(1-\lambda_s) \end{cases}$$

Proof ingredients

$$\left\{\begin{array}{rcl} p-\sum_{i=1}^{s-1}\lambda_i & = & a_1(1-\lambda_1) \\ p-\sum_{i=1}^{s-1}\lambda_i & = & a_2(1-\lambda_2) \\ & \vdots & \\ p-\sum_{i=1}^{s-1}\lambda_i & = & a_{s-1}(1-\lambda_{s-1}) \\ p-\sum_{i=1}^{s-1}\lambda_i & \geqslant & a_s \end{array}\right.$$

Main result in an application oriented version

Theorem

Let $N \geqslant 2$, let $k \geqslant 1$ be an integer. Assume that for some integers $r_1, \ldots, r_{k+1}$ and rational numbers $a_1, \ldots, a_{k+1}$ we have

$$\widehat{\alpha}(\mathbb{P}^{N-1}; r_j) \geqslant a_j \text{ for } j = 1, \ldots, k+1,$$

$$k \leqslant a_j \leqslant k+1 \text{ for } j = 1, \ldots, k, \quad a_1 > k, \quad a_{k+1} \leqslant k+1.$$

Then

$$\widehat{\alpha}(\mathbb{P}^N; r_1 + \ldots + r_{k+1}) \geqslant \left(1 - \sum_{j=1}^{k} \frac{1}{a_j}\right) a_{k+1} + k.$$

Application 1

Proposition

Let s be a positive integer and let k be an integer in the range $1 \leqslant k \leqslant s$. Let Z be a set of

$$r \geqslant r_k = k(s+1)^{N-1} + (s+1-k)s^{N-1}$$

very general points in $\mathbb{P}^N$. Then

$$\widehat{\alpha}(Z) \geqslant s+1-\frac{s+1-k}{s+1}.$$

Application 2

Proposition

The Demailly Conjecture

$$\widehat{\alpha}(I) \geqslant \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1}$$

holds for $s \geqslant m^N$ very general points in $\mathbb{P}^N$.

Application 2

Proposition

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Remark

This improves an earlier result by Malara, Sz. and Szpond, that the Demailly Conjecture holds for $s \geqslant (m+1)^N$ very general points in $\mathbb{P}^N$.