Unexpected hypersurfaces

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Retrospection

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Higher order embeddings

Definition

Let X be a smooth projective variety and L a line bundle on X. For an integer $k \geqslant 0$ we say that L is *k–very ample* if the evaluation map

$$H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective for all 0–dimensional subschemes Z of X of length at most $k + 1$.

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Thus L is

- 0*–very ample if and only if it is* **globally generated**,
- 1*–very ample if and only if it is* **very ample**.

Further developments

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- *a number of other papers due to E. Ballico, Th. Bauer, S. Di Rocco, A. Knutsen, A. Lanteri, S. Rams, W. Syzdek, H. Terakawa, H. Tutaj-Gasińska, C. De Volder, and many others.*

Theorem (Beltrametti, Sz.)

Let X be a smooth Calabi-Yau threefold and let L be an ample and spanned line bundle on X with $L^3 \geqslant \max\{8, k+4\}$ for a positive integer k. Then for an arbitrary partition $k = k_1 + \ldots + k_r$, there is a non-empty Zariski open subset U_r of X^r such that the evaluation map

$$H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_X / I(P_1)^{k_1} \otimes \ldots \times I(P_r)^{k_r})$$

is surjective for $(P_1, \ldots, P_r) \in U_r$.

MPIM Bonn 1998

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Remark

Nowadays this property is known as the **birational k–jet ampleness**.

An "Italian"-type Theorem

Theorem (Beltrametti, Sz.)

Let X be a smooth Calabi-Yau threefold and let L be an ample and spanned line bundle on X. Then nL is k–jet ample, for $k \geqslant 2$, if either

- *$n \geqslant 3k$; or*
- *$k \geqslant 3$, $n \geqslant 2k$ and the image of X under the morphism associated to the complete linear system $|L|$ is not a variety of minimal degree other than $\mathbb{P}^3$; or*
- *$k = 2$, $n \geqslant 5$ and $|L|$ maps X two-to-one onto $\mathbb{P}^3$.*

Resent sample result

Question

Given X and an ample line bundle L determine for each positive integer n the value $\phi(n)$ such that nL is $\phi(n)$–very ample but it is not $(\phi(n)+1)$–very ample.

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Let X be an abelian surface with Picard number one, and let L be an ample generator. Then, for $n \geqslant 2$

$$\phi(n) = (n-1)L^2 - 2.$$

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Theorem (Bauer, Sz. 1997)

For $n = 1$ there is

$$\phi(1) = \left\lfloor \frac{1}{2}\left(\frac{1}{2}L^2 - 3\right)\right\rfloor.$$

General framework

Problem (Postulation problem)

Let X be a smooth projective variety, L be a positive line bundle on X, and Z be a subscheme of X. Determine the value of

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If Z is a set of reduced general points on X, then

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Theorem (Alexander, Hirschowitz)

If Z is a set of reduced general points in $\mathbb{P}^n$, then

$$h^0(\mathbb{P}^n; L \otimes \mathcal{I}_Z^{(2)}) = \max\left\{0, h^0(\mathbb{P}^n; L) - (n+1)|Z|\right\}$$

with a finite list of exceptions.

Fat points

Example

Let P be an arbitrary point in $\mathbb{P}^n$. Then

$$h^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_P^m) = \max\left\{0, \binom{n+d}{n} - \binom{n-1+m}{n}\right\},$$

i.e. *mP imposes independent conditions* in any degree.

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Definition

We say that a set Z of reduced points (not necessarily general) in $\mathbb{P}^n$ admits an *unexpected hypersurface* of degree d with a **general** point P of multiplicity m, if

$$h^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z \otimes \mathcal{I}_P^m) \;>\; \max\left\{0, h^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z) - \binom{n-1+m}{n}\right\}.$$

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Remark

The empty set does not admit any unexpected hypersurfaces.

The first example

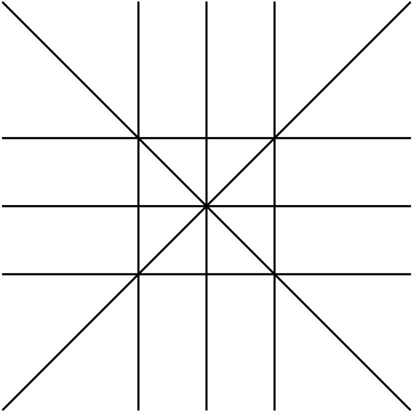
Example

In the article
Di Gennaro, R., Ilardi, G., Vallés, J.: Singular hypersurfaces characterizing the Lefschetz properties. Lond. Math. Soc. 89 (2014) 194–212

the authors observed in passing that

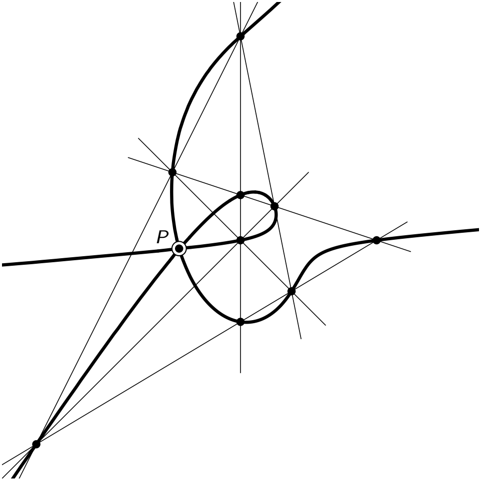
there exists a set Z of 9 points in $\mathbb{P}^2$ (dual to the B_3 arrangement of lines) which admits an unexpected (irreducible) curve of degree 4 (passing through Z) with a general point P of multiplicity 3.

Visualization of B_3



$$xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

Visualization of the unexpected quartic admitted for B_3^*



Equation of the unexpected quartic for B_3^*

Example (Bauer, Malara, Sz., Szpond, arXiv:1804.03610)

Let Z be the set of points with coordinates

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1). \end{array}$$

and let $P = (a:b:c)$ be a general point. Then

$$\begin{aligned} Q_P(x:y:z) = {} & 3a(b^2-c^2)\cdot x^2yz + 3b(c^2-a^2)\cdot xy^2z \\ & + 3c(a^2-b^2)\cdot xyz^2 \\ & + a^3\cdot y^3z - a^3\cdot yz^3 + b^3\cdot xz^3 \\ & - b^3\cdot x^3z + c^3\cdot x^3y - c^3\cdot xy^3 \end{aligned}$$

vanishes at all points of Z and has a triple point in P.

Unexpected duality

Example (Bauer, Malara, Sz., Szpond, arXiv:1804.03610)

For a generic choice of the point $S = (x : y : z)$ the cubic (in variables a, b, c)

$$\begin{aligned} Q_S(a:b:c) = {} & yz(y^2 - z^2)\cdot a^3 + xz(z^2 - x^2)\cdot b^3 \\ & + xy(x^2 - y^2)\cdot c^3 + 3x^2yz\cdot ab^2 \\ & - 3xy^2z\cdot a^2b + 3xyz^2\cdot a^2c - 3x^2yz\cdot ac^2 \\ & + 3xy^2z\cdot bc^2 - 3xyz^2\cdot b^2c \end{aligned}$$

has a triple point in S.

Hence it splits into three lines.

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Remark

This is part of a much more general result due to Harbourne, Migliore, Nagel and Teitler, arXiv:1805.10626. More on this later...

Uniqueness of the B_3^* configuration

Theorem (Farnik, Galuppi, Sodomaco, Trok, arXiv:1804.03590)

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Question

For which d and m are there unexpected hypersurfaces?

Fundamental works

Systematic study of point sets admitting unexpected hypersurfaces has been initiated by D. Cook II, B. Harbourne, J. Migliore and U. Nagel in their article

Line arrangements and configurations of points with an unexpected geometric property, arXiv:1602.02300, to appear in Compositio Math.

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The recent preprint by B. Harbourne, J. Migliore, U. Nagel, and Z. Teitler

Unexpected hypersurfaces and where to find them, arXiv:1805.10626.

brings series of new examples and results.

Duality analysis

Theorem (Harbourne, Migliore, Nadel, Teitler)

Let $Z \subset \mathbb{P}^2$ be a finite set of points admitting an irreducible unexpected curve $C = C_P$ of degree $m+1$ with a general point $P = [a_0 : a_1 : a_2]$ of multiplicity m.

(a) *The curve C_P is unique.*

(b) *Let $F(a, x)$ be the form defining C. Assume that the lines dual to the points of Z comprise a* **free** *line arrangement. Then $F(a, x)$ is bi-homogeneous of bi-degree $(m, m+1)$. Furthermore, viewing $F(a, x)$ as a polynomial in the a variables it has multiplicity m at the general point $[x_0 : x_1 : x_2]$.*

(c) *Assume that $F \in R = \mathbb{C}[a_0, a_1, a_2][x_0, x_1, x_2]$ is any bi-homogeneous form of bi-degree $(m, m+1)$ such that $F(a, x)$ is reduced and irreducible for a general point $a = P$ and has multiplicity m both in the a variables at $a = x$ and in the x variables at $x = a$. Then $F(x, a)$ is the tangent cone at $x = P$ to the curve $F(a, x) = 0$.*

Problem

Do all unexpected curves of degree $d = m + 1$ come from free line arrangements?

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There are examples of non free line arrangements whose dual points admit an unexpected curve, however all such examples found so far can be extended to free arrangements, which give the same unexpected curve!

The connection between the freeness of line arrangements and the existence of unexpected curves is a problem of intensive study (Levico, last week).

A passage to higher dimensions

Definition

A Fermat-type hyperplane arrangement $\mathcal{F}_N^n$ in $\mathbb{P}^N$ is the arrangement determined by linear factors of the polynomial

$$F_{N,n} = \prod_{0 \leqslant i < j \leqslant N} (x_i^n - x_j^n).$$

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Remark

For $N = 2$ we obtain Fermat (Ceva) arrangements of lines defined by

$$(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0.$$

A Fermat-type configuration of points in $\mathbb{P}^3$

We study the ideal I generated by the following 8 binomials of degree 4:

$$x(y^3 - z^3),\; x(z^3 - w^3),\; y(x^3 - z^3),\; y(z^3 - w^3)\,,$$
$$z(x^3 - y^3),\; z(y^3 - w^3),\; w(x^3 - y^3),\; w(y^3 - z^3)\,.$$

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This is the ideal of $27=3^3$ (complete intersection) points of the form

$$P_{(\alpha,\beta,\gamma)}=(1:\varepsilon^\alpha:\varepsilon^\beta:\varepsilon^\gamma)$$

where ε is a primitive root of unity of order 3 and $1\leqslant\alpha,\beta,\gamma\leqslant 3$; and the 4 coordinate points. We denote the set of all these 31 points by W.

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This is a subset of points determined by the arrangement $\mathcal{F}_3^3$.

Unexpected surface in $\mathbb{P}^3$

Theorem (Bauer, Malara, Sz., Szpond)

Let $P = (a : b : c : d)$ be a generic point in $\mathbb{P}^3$. Then the quartic

$$\begin{aligned} Q_R(x:y:z:w) = {} & b^2(c^3-d^3)\cdot x^3y + a^2(d^3-c^3)\cdot xy^3 \\ & + c^2(d^3-b^3)\cdot x^3z + c^2(a^3-d^3)\cdot y^3z \\ & + a^2(b^3-d^3)\cdot xz^3 + b^2(d^3-a^3)\cdot yz^3 \\ & + d^2(b^3-c^3)\cdot x^3w + d^2(c^3-a^3)\cdot y^3w \\ & + d^2(a^3-b^3)\cdot z^3w + a^2(c^3-b^3)\cdot xw^3 \\ & + b^2(a^3-c^3)\cdot yw^3 + c^2(b^3-a^3)\cdot zw^3 \end{aligned}$$

- *vanishes at all points of W,*
- *vanishes to order 3 at P,*
- *is an unexpected surface for W.*

The existence of unexpected hypersurfaces

Theorem (Harbourne, Migliore, Nagel, Teitler)

Denote by d the degree of an unexpected hypersurface of some finite set of points $Z \subset \mathbb{P}^n$ and by m its multiplicity at a general point P in $\mathbb{P}^n$.

(i) *If $n = 2$ then there exists some set Z admitting such an unexpected curve if and only if (d, m) satisfies $d > m > 2$.*

(ii) *If $n \geqslant 3$ then there exists a set Z admitting such an unexpected hypersurface if and only if (d, m) satisfies $d \geqslant m \geqslant 2$.*

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Remark

The construction given by HMNT is explicit and solves the geography problem but we are far from understanding all ways the unexpected hypersurfaces come up.

A sample series of examples

Example (Harbourne, Migliore, Nagel, Teitler)

The root system $B_{n+1} \subset \mathbb{C}^{n+1}$ consists of the $2(n+1)^2$ integer vectors $(a_1, \ldots, a_{n+1})$ such that $1 \leqslant a_1^2 + \cdots + a_{n+1}^2 \leqslant 2$. Thus there is $|Z_{B_{n+1}}| = (n+1)^2$ for the corresponding set of points $Z_{B_{n+1}} \subset \mathbb{P}^n$. These root systems give always rise to unexpected hypersurfaces of degree 4 with a point of multiplicity 4 and sometimes to hypersurfaces with different invariants too.

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Remark

The above claim is, so far, based on computer experiments.

A natural generalization

Question

Can there be an unexpected hypersurface with more than one general fat point?

Higher dimensional projective spaces and more fat points

Theorem (Szpond, on arXiv soon)

Let $N = 2k + 1$ be an odd number. Let W_N be the union of coordinate points in $\mathbb{P}^N$ and the Fermat-type configuration of points

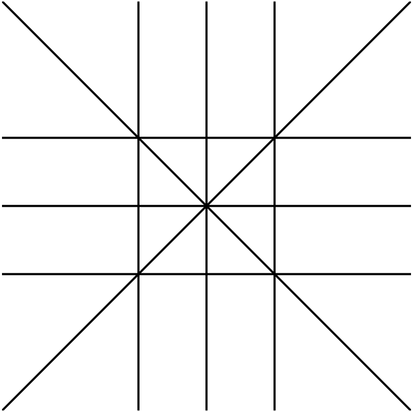
$$(1 : \varepsilon^{\alpha_1} : \varepsilon^{\alpha_2} : \ldots : \varepsilon^{\alpha_N}),$$

where ε is a primitive root of 1 of order 3 and $\alpha_1, \ldots, \alpha_N = 1, 2, 3$. Let R and $P_1, \ldots, P_{k-1}$ be generic points in $\mathbb{P}^N$. Then there exists a **unique** *quartic hypersurface*

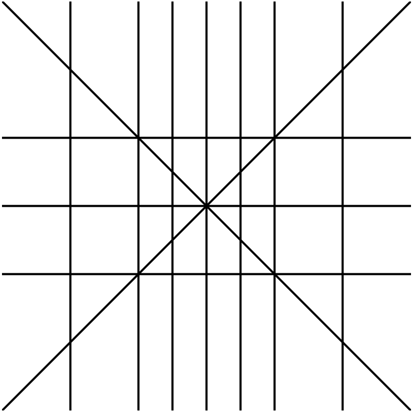
- *vanishing at all points of W_N,*
- *vanishing to order 3 at R,*
- *vanishing to order 2 at $P_1, \ldots, P_{k-1}$,*
- *unexpected for W_N.*

Back to the root

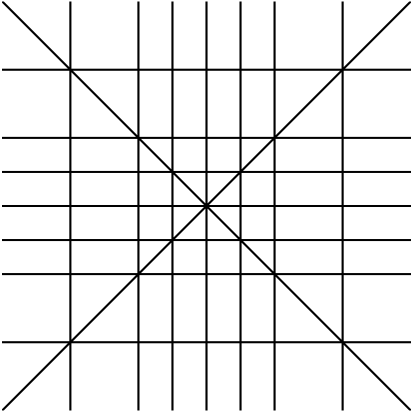
Back to the root system B_3



B_3 extended



B_3 extended and extended



BUON COMPLEANNO MAURO!