Unexpected hypersurfaces

Tomasz Szemberg

Pedagogical University of Cracow
Department of Mathematics

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Problem (Postulation problem)

Let X be a smooth projective variety, L be a positive line bundle on X, and Z be a subscheme of X. Determine the value of

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General framework

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$$h^0(X; L \otimes \mathcal{I}_Z).$$

Example

If Z is a set of reduced **general** points on X, then

$$h^0(X; L \otimes \mathcal{I}_Z) = \max\left\{0, h^0(X; L) - |Z|\right\}.$$

Rising multiplicities to 2 and restricting to $\mathbb{P}^n$

Theorem (Alexander, Hirschowitz)

If Z is a set of s reduced general points on $\mathbb{P}^n$, then

$$h^0(\mathbb{P}^n; \mathcal{O}(d) \otimes \mathcal{I}_Z^{(2)}) = \max\left\{0, h^0(\mathbb{P}^n; \mathcal{O}(d)) - (n+1)s\right\}$$

with a finite list of exceptions:

- $d = 2, 2 \leqslant s \leqslant n$;
- $n = 2, d = 4, s = 5$;
- $n = 3, d = 4, s = 9$;
- $n = 4, d = 4, s = 14$;
- $n = 4, d = 3, s = 7$.

Fat points

Example

Let P be an arbitrary point in $\mathbb{P}^n$. Then

$$h^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_P^m) = \max\left\{0, \binom{n+d}{n} - \binom{n-1+m}{n}\right\},$$

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Definition

We say that a set Z of reduced points (not necessarily general) in $\mathbb{P}^n$ admits an *unexpected hypersurface* of degree d with a **general** point P of multiplicity m, if

$$h^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z \otimes \mathcal{I}_P^m) \; > \; \max\left\{0, h^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z) - \binom{n-1+m}{n}\right\}.$$

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Remark

The empty set does not admit any unexpected hypersurfaces.

Dependence on the underlying variety

Example (Shifrin surface)

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $L = \mathcal{O}(2,1)$. Then

$$h^0(X, L) = 6 \;\text{ but }\; h^0(X, L \otimes I(P)^3) > 0$$

for all points $P \in X$.

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The bundle L is very ample and embeds X to $\mathbb{P}^5$ as a surface of degree 4, which is perfectly **hypo-osculating**, *i.e.,*

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Conjecture (Shifrin)

X is the only smooth perfectly hypo-osculating surface in $\mathbb{P}^5$.

An almost example

Example (Togliatti system)

Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$ be the (bi)rational map

$$(x : y : z) \to (x^2y : x^2z : y^2x : y^2z : z^2x : z^2y).$$

Then the image is a smooth hypo-osculating surface of degree 6.

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The map φ is induced by the linear system $3H - E_1 - E_2 - E_3$, where $f : Y \to \mathbb{P}^2$ is the blow up of the coordinate points P_1, P_2, P_3 with exceptional divisors E_1, E_2, E_3 and $H = f^*(\mathcal{O}(1))$, composed with the projection from the point corresponding to the monomial xyz.

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Remark

Also in this example all unexpected divisors are reducible.

The first example

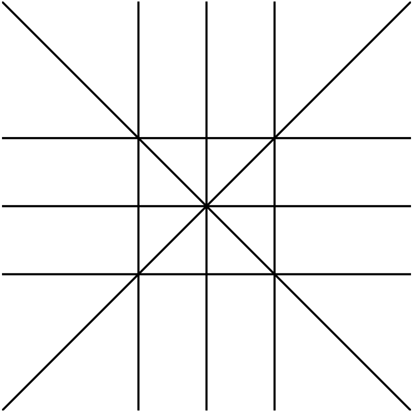
Example

In the article
Di Gennaro, R., Ilardi, G., Vallés, J.: Singular hypersurfaces characterizing the Lefschetz properties. Lond. Math. Soc. 89 (2014) 194–212

the authors observed in passing that

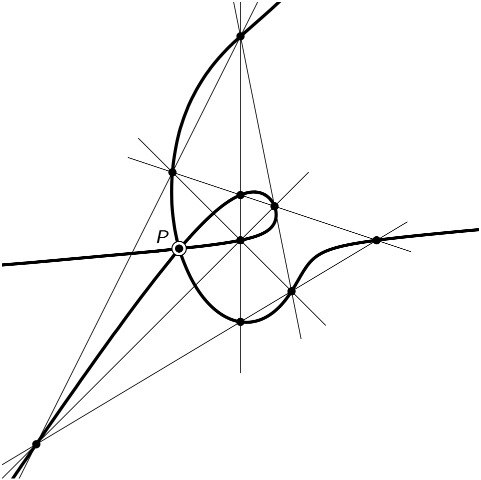
there exists a set Z of 9 points in $\mathbb{P}^2$ (dual to the B_3 arrangement of lines) which admits an unexpected (irreducible) curve of degree 4 (passing through Z) with a general point P of multiplicity 3.

Visualization of B_3



$$xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

Visualization of the unexpected quartic admitted for B_3^*



Equation of the unexpected quartic for B_3^*

Example (Bauer, Malara, Sz., Szpond, arXiv:1804.03610)

Let Z be the set of points with coordinates

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1). \end{array}$$

and let $P = (a:b:c)$ be a general point. Then

$$\begin{aligned} Q_P(x:y:z) = {} & 3a(b^2-c^2)\cdot x^2yz + 3b(c^2-a^2)\cdot xy^2z \\ & + 3c(a^2-b^2)\cdot xyz^2 \\ & + a^3\cdot y^3z - a^3\cdot yz^3 + b^3\cdot xz^3 \\ & - b^3\cdot x^3z + c^3\cdot x^3y - c^3\cdot xy^3 \end{aligned}$$

vanishes at all points of Z and has a triple point in P.

Unexpected duality

Example (Bauer, Malara, Sz., Szpond, arXiv:1804.03610)

For a generic choice of the point $S = (x : y : z)$ the cubic (in variables a, b, c)

$$\begin{aligned} Q_S(a : b : c) = yz(y^2 - z^2) \cdot a^3 + xz(z^2 - x^2) \cdot b^3 \\ + xy(x^2 - y^2) \cdot c^3 + 3x^2yz \cdot ab^2 \\ - 3xy^2z \cdot a^2b + 3xyz^2 \cdot a^2c - 3x^2yz \cdot ac^2 \\ + 3xy^2z \cdot bc^2 - 3xyz^2 \cdot b^2c \end{aligned}$$

has a triple point in S.

Hence it splits into three lines.

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Remark

This is part of a much more general result due to Harbourne, Migliore, Nagel and Teitler, arXiv:1805.10626. More on this later...

Uniqueness of the B_3^* configuration

Theorem (Farnik, Galuppi, Sodomaco, Trok, arXiv:1804.03590)

Up to projective equivalence, the configuration of points B_3^* is the only one which admits an unexpected curve of degree 4.

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Question

For which d and m are there unexpected hypersurfaces?

Fundamental works

Systematic study of point sets admitting unexpected hypersurfaces has been initiated by D. Cook II, B. Harbourne, J. Migliore and U. Nagel in their article

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The recent preprint by B. Harbourne, J. Migliore, U. Nagel, and Z. Teitler

Unexpected hypersurfaces and where to find them, arXiv:1805.10626.

brings series of new examples and results.

Duality analysis

Theorem (Harbourne, Migliore, Nadel, Teitler)

Let $Z \subset \mathbb{P}^2$ be a finite set of points admitting an irreducible unexpected curve $C = C_P$ of degree $m+1$ with a general point $P = [a_0 : a_1 : a_2]$ of multiplicity m.

(a) *The curve C_P is unique.*

(b) *Let $F(a,x)$ be the form defining C. Assume that the lines dual to the points of Z comprise a* **free** *line arrangement. Then $F(a,x)$ is bi-homogeneous of bi-degree $(m, m+1)$. Furthermore, viewing $F(a,x)$ as a polynomial in the a variables it has multiplicity m at the general point $[x_0 : x_1 : x_2]$.*

(c) *Assume that $F \in R = \mathbb{C}[a_0, a_1, a_2][x_0, x_1, x_2]$ is any bi-homogeneous form of bi-degree $(m, m+1)$ such that $F(a,x)$ is reduced and irreducible for a general point $a = P$ and has multiplicity m both in the a variables at $a = x$ and in the x variables at $x = a$. Then $F(x,a)$ is the tangent cone at $x = P$ to the curve $F(a,x) = 0$.*

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Do all unexpected curves of degree $d = m + 1$ come from free line arrangements?

Remark

The answer might depend on the understanding of the word "come".

There are examples of non free line arrangements whose dual points admit an unexpected curve, however all such examples found so far can be extended to free arrangements, which give the same unexpected curve!

A passage to higher dimensions

Definition

A Fermat-type hyperplane arrangement $\mathcal{F}_N^n$ in $\mathbb{P}^N$ is the arrangement determined by linear factors of the polynomial

$$F_{N,n} = \prod_{0 \leqslant i < j \leqslant N} (x_i^n - x_j^n).$$

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Remark

For $N = 2$ we obtain Fermat (Ceva) arrangements of lines defined by

$$(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0.$$

A Fermat-type configuration of points in $\mathbb{P}^3$

We study the ideal I generated by the following 8 binomials of degree 4:

$$x(y^3-z^3),\ x(z^3-w^3),\ y(x^3-z^3),\ y(z^3-w^3),$$

$$z(x^3-y^3),\ z(y^3-w^3),\ w(x^3-y^3),\ w(y^3-z^3).$$

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This is the ideal of $27=3^3$ (complete intersection) points of the form

$$P_{(\alpha,\beta,\gamma)}=(1:\varepsilon^{\alpha}:\varepsilon^{\beta}:\varepsilon^{\gamma})$$

where ε is a primitive root of unity of order 3 and $1\leqslant\alpha,\beta,\gamma\leqslant 3$; and the 4 coordinate points. We denote the set of all these 31 points by W.

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This is a subset of points determined by the arrangement $\mathcal{F}_3^3$.

Theorem (Bauer, Malara, Sz., Szpond)

Let $P = (a : b : c : d)$ be a generic point in $\mathbb{P}^3$. Then the quartic

$$\begin{aligned} Q_R(x:y:z:w) = {} & b^2(c^3-d^3)\cdot x^3y + a^2(d^3-c^3)\cdot xy^3 \\ & + c^2(d^3-b^3)\cdot x^3z + c^2(a^3-d^3)\cdot y^3z \\ & + a^2(b^3-d^3)\cdot xz^3 + b^2(d^3-a^3)\cdot yz^3 \\ & + d^2(b^3-c^3)\cdot x^3w + d^2(c^3-a^3)\cdot y^3w \\ & + d^2(a^3-b^3)\cdot z^3w + a^2(c^3-b^3)\cdot xw^3 \\ & + b^2(a^3-c^3)\cdot yw^3 + c^2(b^3-a^3)\cdot zw^3 \end{aligned}$$

- *vanishes at all points of W,*
- *vanishes to order 3 at P,*
- *is an unexpected surface for W.*

The existence of unexpected hypersurfaces

Theorem (Harbourne, Migliore, Nagel, Teitler)

Denote by d the degree of an unexpected hypersurface of some finite set of points $Z \subset \mathbb{P}^n$ and by m its multiplicity at a general point P in $\mathbb{P}^n$.

(i) *If $n = 2$ then there exists some set Z admitting such an unexpected curve if and only if (d, m) satisfies $d > m > 2$.*

(ii) *If $n \geqslant 3$ then there exists a set Z admitting such an unexpected hypersurface if and only if (d, m) satisfies $d \geqslant m \geqslant 2$.*

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Remark

The construction given by HMNT is explicit and solves the geography problem but we are far from understanding all ways the unexpected hypersurfaces come up.

A sample series of examples

Example (Harbourne, Migliore, Nagel, Teitler)

The root system $B_{n+1} \subset \mathbb{C}^{n+1}$ consists of the $2(n+1)^2$ integer vectors $(a_1, \ldots, a_{n+1})$ such that $1 \leqslant a_1^2 + \cdots + a_{n+1}^2 \leqslant 2$. Thus there is $|Z_{B_{n+1}}| = (n+1)^2$ for the corresponding set of points $Z_{B_{n+1}} \subset \mathbb{P}^n$. These root systems give always rise to unexpected hypersurfaces of degree 4 with a point of multiplicity 4 and sometimes to hypersurfaces with different invariants too.

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Remark

The above claim is, so far, based on computer experiments.

A natural generalization

Question

Can there be an unexpected hypersurface with more than one general fat point?

Higher dimensional projective spaces and more fat points

Theorem (Szpond, on arXiv soon)

Let $N = 2k + 1$ be an odd number. Let W_N be the union of coordinate points in $\mathbb{P}^N$ and the Fermat-type configuration of points

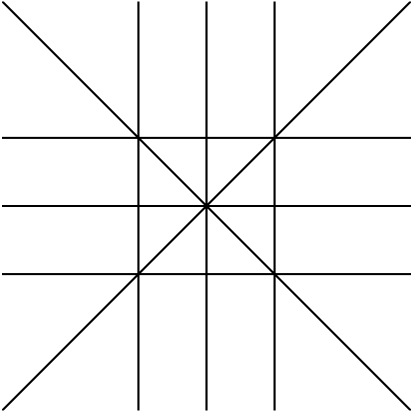
$$(1 : \varepsilon^{\alpha_1} : \varepsilon^{\alpha_2} : \ldots : \varepsilon^{\alpha_N}),$$

where ε is a primitive root of 1 of order 3 and $\alpha_1, \ldots, \alpha_N = 1, 2, 3$. Let R and $P_1, \ldots, P_{k-1}$ be generic points in $\mathbb{P}^N$. Then there exists a **unique** *quartic hypersurface*

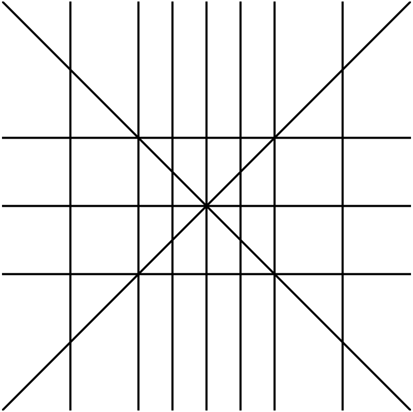
- *vanishing at all points of W_N,*
- *vanishing to order 3 at R,*
- *vanishing to order 2 at $P_1, \ldots, P_{k-1}$,*
- *unexpected for W_N.*

Back to the root

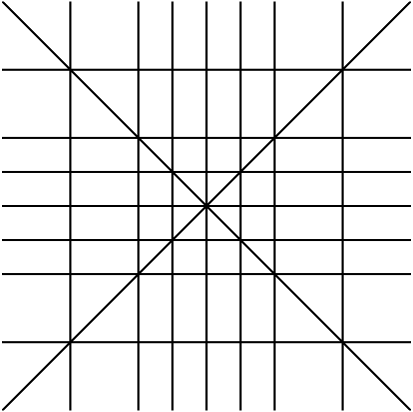
Back to the root system B_3



B_3 extended



B_3 extended and extended



Closing