Postulation in projective spaces and unexpected hypersurfaces

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This talk is based on joint work (Manuscripta) with

Thomas Bauer (Marburg),

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and

Justyna Szpond (PU Cracow).

These slides are available at:
http://szemberg.up.krakow.pl/Cambridge2019.pdf

Definition

Let $I \subset R$ be a homogeneous ideal in a polynomial ring $R = \mathbb{K}[x_0, \ldots, x_N]$. The *Hilbert function* of I is

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Remark

It is well-known that the Hilbert function becomes polynomial, i.e., there is a polynomial $\mathrm{HP}_{R/I}(d)$ *such that*

$$\mathrm{HF}_{R/I}(d) = \mathrm{HP}_{R/I}(d) \quad \textit{for} \quad d \gg 0.$$

Notation in a geometric situation

Remark

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If I is a saturated ideal defining a subscheme $V \subset \mathbb{P}^N(\mathbb{K})$, then we write

$$\mathrm{HF}_V(d) = \mathrm{HF}_{R/I}(d)$$

and

$$\mathrm{HP}_V(d) = \mathrm{HP}_{R/I}(d).$$

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This problem is much too hard in this generality and beyond reach.

The simplest Hilbert functions occur for subvarieties which impose independent (or predictable) conditions on forms of arbitrary degree.

Definition (Carlini, Catalisano, Geramita)

We say that a subscheme $V \subset \mathbb{P}^N(\mathbb{K})$ has a *bipolynomial Hilbert function* if

$$\mathrm{HF}_V(d) = \min\left\{\mathrm{HP}_{\mathbb{P}^N}(d),\ \mathrm{HP}_V(d)\right\}$$

for all d.

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There is

$$\mathrm{HP}_{\mathbb{P}^N}(d) = \binom{N+d}{d}$$

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Theorem (Alexander-Hirschowitz 1995)

Let V be a general collection of s double points in $\mathbb{P}^N(\mathbb{K})$ (over an algebraically closed field of characteristic zero). Then

$$\mathrm{HF}_V(d) = \min\left\{\binom{N+d}{d},\ s(N+1)\right\}$$

except in the following cases

- $d = 2, 2 \le s \le N$;
- $N = 2, d = 4, s = 5$;
- $N = 3, d = 4, s = 9$;
- $N = 4, d = 4, s = 14$;
- $N = 4, d = 3, s = 7$.

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Remark

The authors worked on this problem for over 10 years.

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Remark

All proofs are based on some **degeneration***, i.e., if the claim holds for points in* **special** *position, then it holds for points in* **general** *position (provided both positions belong to a flat family).*

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Remark

Ciliberto and Miranda introduced a **degeneration** *of the ambient space (replace* $\mathbb{P}^2$ *by some other scheme) combined with the degeneration of points.*

Conjecture (SHGH, Segre-Harbourne-Gimigliano-Hirschowitz)

Let V be a collection of s general points of multiplicity m in $\mathbb{P}^2(\mathbb{K})$. Then either

$$\mathrm{HF}_V(d) = \min\left\{ \binom{d+2}{2}, \ s\binom{m+1}{2} \right\}$$

or the linear system

$$|\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}_V^{(m)}|$$

contains a fat (-1)-curve in its base locus.

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Conjecture (Nagata 1959)

Let V be a collection of $s \geq 10$ general points of multiplicity m in $\mathbb{P}^2(\mathbb{K})$. Then **the initial degree** *d of $I(V)$ satisfies*

$$d > m\sqrt{s}.$$

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Remark

This is well-known if s is a perfect square and open otherwise.

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Definition

We say that a set Z of reduced points (not necessarily general) in $\mathbb{P}^n$ admits an *unexpected hypersurface* of degree d with a **general** point P of multiplicity m, if

$$h^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z \otimes \mathcal{I}_P^m) \; > \; \max\left\{0, h^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z) - \binom{n-1+m}{n}\right\}.$$

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Remark

The empty set does not admit any unexpected hypersurfaces.

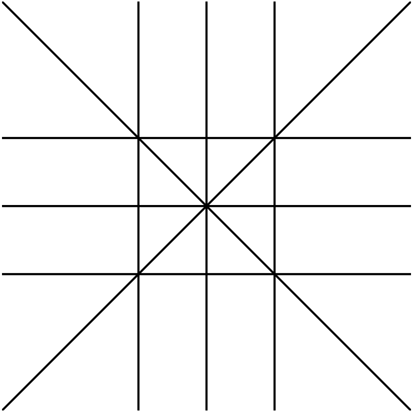
Example

In the article
Di Gennaro, R., Ilardi, G., Vallés, J.: Singular hypersurfaces characterizing the Lefschetz properties. Lond. Math. Soc. 89 (2014) 194–212

the authors observed in passing that

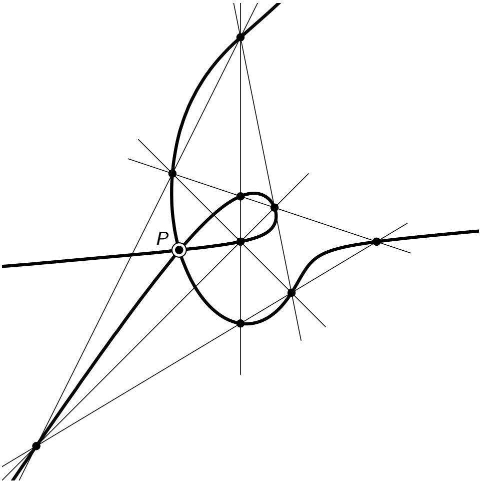
there exists a set Z of 9 points in $\mathbb{P}^2$ (coming from the B_3 root system) which admits an unexpected (irreducible) curve of degree 4 (passing through Z) with a general point P of multiplicity 3.

A visualization of the dual B_3 line arrangement



$$xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

Visualization of the unexpected quartic admitted for B_3



Example (Bauer, Malara, Sz., Szpond, Manuscripta 2019)

Let Z be the set of points with coordinates

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1). \end{array}$$

and let $P = (a:b:c)$ be a general point. Then

$$\begin{aligned} Q_P(x:y:z) \quad = \quad & 3a(b^2-c^2)\cdot x^2yz + 3b(c^2-a^2)\cdot xy^2z \\ & +3c(a^2-b^2)\cdot xyz^2 \\ & +a^3\cdot y^3z - a^3\cdot yz^3 + b^3\cdot xz^3 \\ & -b^3\cdot x^3z + c^3\cdot x^3y - c^3\cdot xy^3 \end{aligned}$$

vanishes at all points of Z and has a triple point at P.

Example (Bauer, Malara, Sz., Szpond, Manuscripta 2019)

For a generic choice of the point $S = (x : y : z)$ the cubic (in variables a, b, c)

$$\begin{aligned} Q_S(a:b:c) &= yz(y^2 - z^2)\cdot a^3 + xz(z^2 - x^2)\cdot b^3 \\ &\quad + xy(x^2 - y^2)\cdot c^3 + 3x^2yz\cdot ab^2 \\ &\quad - 3xy^2z\cdot a^2b + 3xyz^2\cdot a^2c - 3x^2yz\cdot ac^2 \\ &\quad + 3xy^2z\cdot bc^2 - 3xyz^2\cdot b^2c \end{aligned}$$

has a triple point in S.

Hence it splits into three lines.

Theorem (Farnik, Galuppi, Sodomaco, Trok, arXiv:1804.03590)

Up to projective equivalence, the configuration of points B_3 is the only one which admits an unexpected curve of degree 4.

Uniqueness of the B_3 related quartic

Theorem (Farnik, Galuppi, Sodomaco, Trok, arXiv:1804.03590)

Up to projective equivalence, the configuration of points B_3 is the only one which admits an unexpected curve of degree 4.
There is no unexpected curve of degree $d \leq 3$ (this reproves a result from S. Akesseh thesis, Lincoln 2017).

Question

For which d and m are there unexpected hypersurfaces?

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Theorem (Harbourne, Migliore, Nagel, Teitler)

Denote by d the degree of an unexpected hypersurface of some finite set of points $Z \subset \mathbb{P}^n$ and by m its multiplicity at a general point P in $\mathbb{P}^n$.

(i) *If $n = 2$ then there exists some set Z admitting such an unexpected curve if and only if (d, m) satisfies $d > m > 2$.*

(ii) *If $n \geqslant 3$ then there exists a set Z admitting such an unexpected hypersurface if and only if (d, m) satisfies $d \geq m \geq 2$.*

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(ii) *If $n \geqslant 3$ then there exists a set Z admitting such an unexpected hypersurface if and only if (d, m) satisfies $d \geq m \geq 2$.*

Remark

The construction given by HMNT is explicit and solves the geography problem but we are far from understanding all ways the unexpected hypersurfaces come up.

Definition

A Fermat-type hyperplane arrangement $\mathcal{F}_N^n$ in $\mathbb{P}^N$ is the arrangement determined by linear factors of the polynomial

$$F_{N,n} = \prod_{0 \leq i < j \leq N} (x_i^n - x_j^n).$$

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Remark

For $N = 2$ we obtain Fermat (Ceva) arrangements of lines defined by the equation

$$(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0.$$

We study the ideal I generated by the following 8 binomials of degree 4:

$$x(y^3 - z^3),\ x(z^3 - w^3),\ y(x^3 - z^3),\ y(z^3 - w^3),$$

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This is the ideal of $27 = 3^3$ (complete intersection) points of the form

$$P_{(\alpha,\beta,\gamma)} = (1 : \varepsilon^\alpha : \varepsilon^\beta : \varepsilon^\gamma)$$

where ε is a primitive root of unity of order 3 and $1 \leqslant \alpha, \beta, \gamma \leqslant 3$; and the 4 coordinate points. We denote the set of all these 31 points by W.

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This is a subset of points determined by the arrangement $\mathcal{F}_3^3$.

Theorem (Bauer, Malara, Sz., Szpond)

Let $P = (a : b : c : d)$ be a generic point in $\mathbb{P}^3$. Then the quartic

$$
\begin{aligned}
Q_R(x:y:z:w) \quad = \quad & b^2(c^3-d^3)\cdot x^3y + a^2(d^3-c^3)\cdot xy^3 \\
& +c^2(d^3-b^3)\cdot x^3z + c^2(a^3-d^3)\cdot y^3z \\
& +a^2(b^3-d^3)\cdot xz^3 + b^2(d^3-a^3)\cdot yz^3 \\
& +d^2(b^3-c^3)\cdot x^3w + d^2(c^3-a^3)\cdot y^3w \\
& +d^2(a^3-b^3)\cdot z^3w + a^2(c^3-b^3)\cdot xw^3 \\
& +b^2(a^3-c^3)\cdot yw^3 + c^2(b^3-a^3)\cdot zw^3
\end{aligned}
$$

- *vanishes at all points of W,*
- *vanishes to order 3 at P,*
- *is an unexpected surface for W.*

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Can there be an unexpected hypersurface with more than one general fat point?

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Theorem (Szpond, arXiv:1812.04032)

Let $N = 2k + 1$ be an odd number. Let W_N be the union of coordinate points in $\mathbb{P}^N$ and the Fermat-type configuration of points

$$(1 : \varepsilon^{\alpha_1} : \varepsilon^{\alpha_2} : \ldots : \varepsilon^{\alpha_N}),$$

where ε is a primitive root of 1 of order 3 and $\alpha_1, \ldots, \alpha_N = 1, 2, 3$. Let R and $P_1, \ldots, P_{k-1}$ be generic points in $\mathbb{P}^N$. Then there exists a **unique** *quartic hypersurface*

- *vanishing at all points of W_N,*
- *vanishing to order 3 at R,*
- *vanishing to order 2 at $P_1, \ldots, P_{k-1}$,*
- *unexpected for W_N.*

Example (Harbourne, Migliore, Nagel, Teitler)

The root system $B_{n+1} \subset \mathbb{C}^{n+1}$ consists of the $2(n+1)^2$ integer vectors $(a_1, \dots, a_{n+1})$ such that

$$1 \leq a_1^2 + \dots + a_{n+1}^2 \leq 2.$$

Thus there is $|Z_{B_{n+1}}| = (n+1)^2$ for the corresponding set of points $Z_{B_{n+1}} \subset \mathbb{P}^n$.
These root systems give always rise to unexpected hypersurfaces of degree 4 with a point of multiplicity 4 and sometimes to hypersurfaces with different invariants too.

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Remark

The above claim is, so far, based on computer experiments.

Example (Harbourne, Migliore, Nagel, Teitler)

Let $Z \subset \mathbb{P}^3$ be defined by the F_4 root system:

$$(1:0:0:0), (0:1:0:0), (0:0:1:0), (0:0:0:1),$$

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Then Z admits an unexpected surface of degree 4 and a general point R of multiplicity 4.

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Then Z admits an unexpected surface of degree 4 and a general point R of multiplicity 4.

Proposition (Chiantini, Migliore)

Projecting Z from R, one gets a complete intersection.

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