Introduction to projective varieties

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This is still probably far from being a final version, especially since I had no time yet to complete the second part (which is so far not well connected with the first one). Anyway, any kind of comments are very welcome, in particular those concerning the general structure, or suggestions for shorter and/or correct proofs.

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The scope of these notes is to present a soft and practical introduction to algebraic geometry, i.e. with very few algebraic requirements but arriving soon to deep results and concrete examples that can be obtained "by hand". The notes are based on some basic PhD courses (Milan 1998 and Florence 2000) and a summer course (Perugia 1998) that I taught. I decided to produce these notes while preparing new similar courses (Milan and Perugia 2001).

My approach consists of avoiding all the algebraic preliminaries that a standard algebraic geometry course uses for affine varieties and thus start directly with projective varieties (which are the varieties that have good properties). The main technique I use is the Hilbert polynomial, from which it is possible to rigorously and intuitively introduce all the invariants of a projective variety (dimension, degree and arithmetic genus). It is also possible to easily prove the projective Nullstellensatz (from which the standard affine Nullstellensatz can in fact be obtained).

The price to pay for this shortcut is that the way to produce the important results (the most important one for practical purposes is the theorem about the dimension of the fibers) is not always clear, since many results or even definitions have local nature. This was in fact the eventual motivation to write these notes, to show that it is possible to follow such a risky path in a coherent way. Moreover, if the students of a course have all the delicate steps written down, it is possible for the teacher to avoid the too technical results and concentrate on examples and intuitive results.

If the goal of theses notes is achieved, an interested student with very small knowledge of commutative algebra (a sight to Chapter 0, devoted to preliminaries should be enough to figure out the required background) should be able to acquire enough techniques to manipulate varieties and families and compute their dimensions. And if the student becomes interested, he/she could then follow a more advanced text or course.

The present version contains the beginning of a second part which, if ever finished, will eventually contain a first introduction to the theory of schemes.

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0. Algebraic background

We will recall in this chapter the main algebraic ingredients that the reader is assumed to know as a minimum background for the rest of the notes. We will just give the appropriate definitions, state the results and leave the proofs (for those for whom they are new concepts) as exercises, as long as it looks reasonable to do so.

The key results the reader should know about ideals are collected in the following exercise.

Exercise 0.1. Let R be a (commutative and unitary) ring and I an ideal.

(i) If I is maximal then I is prime.

(ii) The set $\sqrt{I} := \{F \in R \mid F^d \in I \text{ for some } d \in \mathbb{N}\}$ is an ideal of R.

(iii) If I is prime, then it is radical (i.e. $\sqrt{I} = I$).

(iv) If $I = I_1 \cap \ldots \cap I_n$ then $\sqrt{I} = \sqrt{I_1} \cap \ldots \cap \sqrt{I_n}$.

(v) If I is prime and $I_1 \cap \ldots \cap I_n \subset I$ then $I_i \subset I$ for some $i = 1, \ldots, n$.

(vi) If I is an ideal contained in a finite union of prime ideals, then I is contained in one of those prime ideals.

(vii) If I is any ideal of R, then $\sqrt{I}$ is the intersection of the prime ideals containing I. [Hint: If $f \notin \sqrt{I}$, use Zorn's Lemma to find a maximal element of the set of ideals $J \supset I$ such that $f \notin \sqrt{J}$, and prove that such a maximal element is a prime ideal]

(viii) If R' is another ring, $f : R' \to R$ is a ring homomorphism (we will always assume that a ring homomorphism sends the unit element of R' to the unit element of R) and I is a prime ideal of R, then $f^{-1}(I)$ is a prime ideal of R'.

Definition. A *primary ideal* of a ring R is an ideal I with the property that if $FG \in I$ but $G \notin I$ then there exists some $d \in \mathbb{N}$ such that $F^d \in I$. It is immediate to see that if I is primary, then $P := \sqrt{I}$ is a prime ideal. The ideal I is then said to be P-*primary.*

Exercise 0.2. Let I be an ideal of a ring R.

(i) Prove that if I is primary then $\sqrt{I}$ is prime.

(ii) Find a counterexample showing that it is not true that an ideal whose radical is prime is necessarily primary (the reader should be able to produce many examples after section 2).

(iii) If $\sqrt{I}$ is a maximal ideal, prove that I is a primary ideal.

(iv) If $I = \cap_i I_i$ where each I_i is a P-primary ideal, then I is also P-primary.

(v) If R is a polynomial ring and I is generated by f^m, f being an irreducible polynomial, then I is $(f) - primary$.

(vi) If R' is another ring, $f : R' \to R$ is a ring homomorphism and I is P-primary, then $f^{-1}(I)$ is $f^{-1}(P)$-primary (observe that from Exercise 0.1(viii) $f^{-1}(P)$ is a prime ideal).

We are going to work with polynomial rings over a field. The main result about these rings is that all their ideals can be generated by a finite number of elements. This will be a consequence of the so-called Hilbert's bases theorem, which we will prove below.

Definition. A ring R is called a *noetherian ring* if any ideal of I admits a finite number of generators, or equivalently if R does not contain an infinite strictly ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq \dots$.

Exercise 0.3. Prove that indeed the above two definitions are equivalent. [Hint: Observe that the union of all the ideals in an ascending chain is an ideal].

Theorem 0.4 (Hilbert's basis theorem)**.** *Let R be a noetherian ring. Then the polynomial ring $R[X]$ is noetherian.*

Proof: Let I be an ideal of $R[X]$. We can assume $I \neq R[X]$, since otherwise 1 would be a generator of I. For each $d \in \mathbb{N}$, the set $J_d := \{r \in R \mid r$ is the leading coefficient of some polynomial of degree d in $I\}$ is easily seen to be an ideal of R (if we take the convention that $0 \in J_d$) and $J_1 \subset J_2 \subset \dots$. Since R is noetherian, there exists $d_0 \in \mathbb{N}$ such that $J_d = J_{d_0}$ if $d \geq d_0$. On the other hand, we can find polynomials $f_1, \dots, f_m \in I$ such that each $J_0, \dots, J_{d_0}$ is generated by the leading coefficients of some (not necessarily all) of these polynomials. Let us see that these polynomials generate I.

Take $f \in I$ and let d be its degree. Assume first that $d \geq d_0$. Then the leading coefficient of f is a linear combination (with coefficients in R) of the leading coefficients of $f_1, \dots, f_m$. Multiplying each f_i by $X^{d-\deg f_i}$ we see that there exist monomials $h_1, \dots, h_n \in R[X]$ such that $f - h_1 f_1 - \dots - h_n f_n$ (which is still in I) has degree strictly less than d. Iterating the process we arrive to $g_1, \dots, g_n \in R[X]$ such that $f - g_1 f_1 - \dots - g_n f_n$ has degree strictly less that d_0. Hence we can assume $d < d_0$. But since now J_d is generated by some leading coefficients of $f_1, \dots, f_n$, we can find $r_1, \dots, r_n \in R$ such that $f - r_1 f_1 - \dots - r_n f_n$ has degree strictly smaller than d. Iterating the process till degree zero we then find that it is possible to write f as a linear combination of $f_1, \dots, f_n$, which concludes the proof. □

Exercise 0.5. Prove a stronger result in case R is a field, namely that any ideal in $\mathbb{K}[X]$ is principal, i.e. generated by one polynomial (a ring with this property is called a *principal ideal domain* or *PID* for short). [Hint: Consider a nonzero polynomial of minimum degree of an ideal and prove, dividing by it, that any other polynomial of the ideal is a multiple of it].

Exercise 0.6. Prove, using induction on n and the Hilbert's basis theorem, that the polynomial ring $\mathbb{K}[X_1,\ldots,X_n]$ is noetherian.

Exercise 0.7. Prove that, for any ideal $I\subset\mathbb{K}[X_1,\ldots,X_n]$, there exists some $m\in\mathbb{N}$ such that $\left(\sqrt{I}\right)^m\subset I$ (in fact this is true for any ideal of a noetherian ring).

Since we are going to work with projective varieties, we will necessarily work with homogeneous polynomials. Let us briefly recall the main definitions and results that we will use.

Definition. A *graded ring* is a ring S such that, as an additive group, is a direct sum $S=\bigoplus_{d\geq 0}S_d$, and the multiplication in S is compatible with the degree in the following sense: if $F\in S_d$ and $G\in S_e$ then $FG\in S_{d+e}$. A *homogeneous element* of S is an element of some $_d$, and d is called the *degree of the element*. Given any $F\in S$, it is possible to write it in a unique way as $F=F_r+\ldots+F_d$, with each F_i in S_i and $F_r,F_d\neq 0$. The nonzero elements among $F_r,\ldots,F_d$ are called the *homogeneous components* of F.

All the graded rings we will deal with throughout these notes will be obtained from the simple example in which $S=\mathbb{K}[X_0,\ldots,X_n]$ ($\mathbb{K}$ being a field) and S_d is the set of homogeneous polynomials of degree d (including also the zero polynomial).

Definition. A *homogeneous ideal* of a graded ring S is an ideal $I\subset S$ such that for any $F\in I$ it holds that all the homogeneous components of F belong to I.

The main properties about graded rings and ideals, left as an exercise, are collected in the following exercise.

Exercise 0.8. Let S be a graded ring and let $I\subset S$ be an ideal.

(i) The ideal I is homogeneous if and only if I is generated by homogeneous elements.

(ii) If S is noetherian and I is homogeneous then I is generated by a finite number of homogeneous elements.

(iii) If I is homogeneous then the quotient S/I has a natural structure of graded ring.

(iv) If I is homogeneous then it is prime if and only if for any $F,G\in S$ homogeneous such that $FG\in I$ it holds that either $F\in I$ or $G\in I$.

(v) If I is homogeneous then $\sqrt{I}$ is also homogeneous.

(vi) If I is homogeneous then it is primary if and only if for any homogeneous $F\in S$ such that $FG\in I$ it holds that either $F\in\sqrt{I}$ or $G\in I$.

(vii) If I is homogeneous then it is radical if and only if for any homogeneous $F \in S$ such that $F^e \in I$ for some $e \in \mathbb{N}$ it holds that $F \in I$.

[Hint: the standard trick is to assume that some suitable element does not belong to the appropriate ideal and then take the homogeneous component of smallest degree not belonging to the ideal].

We will also need the following generalization.

Definition. A *graded module* over a graded ring S is a module M over S such that, as a group M decomposes as a direct sum $\bigoplus_{d\geq 0} M_d$ such that if $F \in S_d$ and $m \in M_e$ then $Fm \in M_{d+e}$. A *graded homomorphism* between two graded modules M and M' is an S-homomorphism $f : M \to M'$ such that $f(M_d) \subset M'_d$ for any $d \in \mathbb{N}$.

Notation. For any $a \in \mathbb{N}$, $M(-a)$ will denote the graded S-module that, as a set, coincides with the graded module M, but for which however we will take as homogeneous part of degree d the homogeneous part of degree $d - a$ of M, i.e. M_{d-a}. In this way, for instance the multiplication by a homogeneous element $F \in S_d$ is a graded homomorphism between $M(-d)$ and M.

1. Projective sets and their ideals; Weak Nullstellensatz

General notation. We fix a ground field $\mathbb{K}$, which we will always assume to be algebraically closed (we will nevertheless recall this fact in the statement of the main theorems). Let $\mathbb{P}^n$ denote the projective space of lines in the vector space $\mathbb{K}^{n+1}$. An element of $\mathbb{P}^n$ represented by the line generated by the nonzero vector $v = (a_0, \ldots, a_n) \in \mathbb{K}^{n+1}$ will be denoted by $[v] = (a_0 : \ldots : a_n)$. When no confusion will arise, we will write just S for the graded ring $\mathbb{K}[X_0, \ldots, X_n]$. The maximal ideal $(X_0, \ldots, X_n)$ will be denoted by $\mathfrak{M}$, and often called *the irrelevant ideal.*

Let $F \in \mathbb{K}[X_0, \ldots, X_n]$ be a polynomial of degree d with homogeneous decomposition $F = F_0 + \ldots + F_d$. Given a point $a = (a_0 : \ldots : a_n) \in \mathbb{P}^n$, we cannot define the expression $F(a)$ as $F(a_0, \ldots, a_n)$, since it clearly depends on the choice of a vector representing a. Indeed, a general representative for a will have the form $(\lambda a_0, \ldots, \lambda a_n)$ (with $\lambda \neq 0$) and then $F((\lambda a_0, \ldots, \lambda a_n)) = F_0(\lambda a_0, \ldots, \lambda a_n) + \ldots F(\lambda a_0, \ldots, \lambda a_n) = F_0(a_0, \ldots, a_n) + \ldots + \lambda^d F_d(a_0, \ldots, a_n)$, which clearly varies when λ varies. However, if F is homogeneous of degree d, we have $F(\lambda a_0, \ldots, \lambda a_n) = \lambda^d F(a_0, \ldots, a_n)$. Even if then $F(a)$ is not defined neither, it makes sense at least to say when it is zero, since obviously $F(\lambda a_0, \ldots, \lambda a_n) = 0$ for any $\lambda \neq 0$ if and only if $F(a_0, \ldots, a_n)$. The main objects we are going to study will be the subsets of a projective space defined as zeros of homogeneous polynomials. More precisely:

Definition. A *projective set* $X \subset \mathbb{P}^n$ is a subset for which there exists a set of homogeneous polynomials $\{F_j \mid j \in J\}$ such that $X = \{p \in \mathbb{P}^n \mid F_j(p) = 0 \text{ for all } j \in J\}$.

For practical reasons, and in view of the previous observation, we will say that $F(a) = 0$ for a point $a \in \mathbb{P}^n$ and an arbitrary polynomial $F \in \mathbb{K}[X_0, \ldots, X_n]$ if and only if any homogeneous component of F vanishes at a. With this convention we can make the following definitions:

Definition. The *projective set defined by a subset* $T \subset \mathbb{K}[X_0, \ldots, X_n]$ will be $V(T) = \{a \in \mathbb{P}^n \mid F(a) = 0 \text{ for any } F \in T\}$. *The homogenous ideal of a subset* $X \subset \mathbb{P}^n$ will be the ideal $I(X) = \{F \in \mathbb{K}[X_0, \ldots, X_n] \mid F(a) = 0 \text{ for any } a \in X\}$. The *graded ring of a projective set* X is the ring $S(X) = \mathbb{K}[X_0, \ldots, X_n]/I(X)$.

Proposition 1.1. *The operators V and I satisfy the following properties:*

(i) $I(\mathbb{P}^n) = \{0\}$ *(for this we just need $\mathbb{K}$ to be infinite),* $I(\emptyset) = \mathbb{K}[X_0, \ldots, X_n]$, $V(\{0\}) = \mathbb{P}^n$, *and* $V(\{1\}) = \emptyset$.

(ii) If $T \subset \mathbb{K}[X_0, \ldots, X_n]$ *and* $< T >$ *is the ideal generated by* T, *then* $V(T) = V(< T >)$. *In particular, any projective set can be defined by a finite number of equations.*

(iii) If $T \subset T' \subset \mathbb{K}[X_0, \ldots, X_n]$, *then* $V(T') \subset V(T) \subset \mathbb{P}^n$.

(iv) If $\{T_j\}_{j\in J}$ is a collection of subsets of $\mathbb{K}[X_0,\dots,X_n]$ then $V(\bigcup_{j\in J} T_j)=\bigcap_{j\in J} V(T_j)$.

(v) If $\{I_j\}_{j\in J}$ is a collection of ideals of $\mathbb{K}[X_0,\dots,X_n]$ then $V(\sum_{j\in J} I_j)=\bigcap_{j\in J} V(I_j)$.

(vi) If $I\subset\mathbb{K}[X_0,\dots,X_n]$ is any homogeneous ideal, then $V(I)=V(\sqrt{I})$.

(vii) If $I,I'\subset\mathbb{K}[X_0,\dots,X_n]$ are two homogeneous ideals, then $V(I\cap I')=V(II')=V(I)\cup V(I')$.

(viii) For any $X\subset\mathbb{P}^n$, $I(X)$ is a homogeneous radical ideal. If X is a projective set, $I(X)$ is the maximum ideal defining X.

(ix) If $X\subset X'\subset\mathbb{P}^n$ then $I(X')\subset I(X)$.

(x) If $\{X_j\}_{j\in J}$ is a collection of subsets of $\mathbb{P}^n$, then $I(\bigcup_{j\in J} X_j)=\bigcap_{j\in J} I(X_j)$.

(xi) For any $X\subset\mathbb{P}^n$, $X\subset VI(X)$, with equality if and only if X is a projective set. In particular $VI(X)$ is the minimum projective set containing X.

(xii) For any $T\subset\mathbb{K}[X_0,\dots,X_n]$, $T\subset IV(T)$ and $VIV(T)=V(T)$.

Proof: We will just prove the first part of (i), leaving the rest as an easy exercise. So we just need to prove that any homogeneous polynomial vanishing at $\mathbb{P}^n$ is the zero polynomial. We will prove it by induction on n, the case $n=0$ being trivial. So assume $n>1$ and write $F=A_0+A_1X_n+\dots A_dX_n^d$, with $A_0,A_1,\dots,A_d\in\mathbb{K}[X_0,\dots,X_{n_1}]$ and $A_d\neq 0$. We thus know by induction hypothesis that we can find $(a_0:\dots:a_{n-1})$ such that $A_d(a_0,\dots,a_{n-1})\neq 0$. But then the polynomial $F(a_0,\dots,a_{n-1},X_n)\in\mathbb{K}[X_n]$ is nonzero, so it has a finite number of roots. Hence the fact that $\mathbb{K}$ in infinite implies that we can find a point $(a_0:\dots:a_{n-1}:a_n)$ not vanishing on F. □

Definition. Part (i), (iv) and (vii) of Proposition 1.1 show that the set of projective sets satisfy the axioms to be the closed sets of a topology in $\mathbb{P}^n$. This topology in which the closed sets are exactly the projective sets is called the *Zariski topology* on $\mathbb{P}^n$. The intersection of a projective set with an open set will be called a *quasiprojective set.* The topology induced by the Zariski topology on any quasiprojective set will be still called Zariski topology on that quasiprojective variety.

Exercise 1.2. Prove that a basis for the Zariski topology on $\mathbb{P}^n$ is given by the open sets of the form $D(F)=\mathbb{P}^n\setminus V(F)$, where F is a homogeneous polynomial.

We list now a series of examples of projective sets.

Example 1.3. Any linear subspace Λ of $\mathbb{P}^n$ is clearly a projective set, since it is defined by homogeneous linear forms. In fact, the homogeneous ideal of Λ is generated by any set of linear equations defining Λ. Indeed we can assume, after changing coordinates, that Λ is defined by the equations $X_{r+1} = \ldots = X_n = 0$. Thus we write any homogeneous homogeneous polynomial $F \in I(\Lambda)$ in the form $F = X_{r+1}G_{r+1} + \ldots + X_nG_n + H(X_0, \ldots, X_r)$. Since H belongs to $I(\Lambda)$ (because $F, X_{r+1}, \ldots, X_n$ do), then it follows that $H(a_0, \ldots, a_r) = 0$ for any $a_0, \ldots, a_n \in \mathbb{K}$, which implies that H is the zero polynomial. Therefore $F \in (X_{r+1}, \ldots, X_n)$ and thus $X_{r+1}, \ldots, X_n$ generate the ideal $I(\Lambda)$. Observe in particular that $S(\Lambda) \cong \mathbb{K}[X_0, \ldots, X_r]$ as graded rings. As a particular case of the above, any point of $\mathbb{P}^n$ is a projective set. Therefore, by Proposition 1.1(vii), also any finite set of points is a projective set.

Exercise 1.4. Prove that the homogeneous ideal of the set $\{(1:0:0), (0:1:0)\} \subset \mathbb{P}^2$ is (X_2, X_0X_1). Interpret geometrically this result and find then the homogeneous ideal of the set $\{(2:3:-1), (1:-2:2)\}$. Try to generalize the exercise to sets of three and four points in $\mathbb{P}^2$ (now you will need to take care of the possible different configurations of the points).

Example 1.5. Identify $\mathbb{P}^{nm+m+n}$ with the set of nonzero $(n+1)\times(m+1)$-matrices modulo scalars. Then the set of matrices of rank at most k is a projective set, since it defined by the vanishing of all the minors of order $k+1$, which are homogeneous polynomials of degree $k+1$ in the variables of $\mathbb{P}^{nm+m+n}$. It is not at all trivial that the homogeneous ideal of this projective set is generated by all these minors.

Exercise 1.6. Prove that, in the example above, the projective set of matrices of rank one is in bijection with $\mathbb{P}^n \times \mathbb{P}^m$ via the map $\varphi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+m+n}$ (well) defined by

$$\varphi_{n,m}((X_0, \ldots, X_n), (Y_0 : \ldots : Y_m)) = \begin{pmatrix} X_0Y_0 & \ldots & X_0Y_m \\ \vdots & \ddots & \vdots \\ X_nY_0 & \ldots & X_nY_m \end{pmatrix}$$

Prove also that $\varphi_{n,m}$ maps any $\mathbb{P}^n \times \{q\}$ and any $\{p\} \times \mathbb{P}^m$ in linear subspaces of $\mathbb{P}^{nm+m+n}$, and the same changing $\mathbb{P}^n$ or $\mathbb{P}^m$ by any linear subspace of them.

Definition. The above map $\varphi_{n,m}$ is called the *Segre embedding* of $\mathbb{P}^n \times \mathbb{P}^m$, and its image is called the *Segre variety.*

Example 1.7. In a similar way, we can identify $\mathbb{P}^{\binom{n+2}{2}-1} = \mathbb{P}^{\frac{n(n+3)}{2}}$ with the set of symmetric $(n+1)\times(n+1)$-matrices modulo scalars and the set of matrices of rank at most k is a projective set. It is again a non-trivial fact that its homogeneous ideal is generated by the homogeneous polynomials of degree $k+1$ defined by all the $(k+1)\times(k+1)$-minors. In the case $k=1$, identifying a symmetric matrix with the quadric it defines, we can also regard the projective set of matrices of rank one as the set of double hyperplanes inside the set of all quadrics in $\mathbb{P}^n$. To be nasty, this identification of quadrics and matrices is only valid if char$k \neq 2$. And observe also that the point in $\mathbb{P}^{\binom{n+2}{2}-1}$ corresponding to a quadric defined by $F = \sum_{i,j} a_{ij} X_i X_j$ has not coordinates a_{ij}, but $\frac{1}{2} \frac{\partial^2 F}{\partial X_i \partial X_j}$ (which coincides with a_{ij} only if $i = j$; otherwise it is $\frac{1}{2} a_{ij}$).

Exercise 1.8. Prove that, in the example above, the projective set of matrices of rank one is in bijection with $\mathbb{P}^n$ via the map $\nu_2 : \mathbb{P}^n \to \mathbb{P}^{\binom{n+2}{2}-1}$ defined by

$$\nu_2(a_0 : \ldots : a_n) = \begin{pmatrix} a_0^2 & \ldots & a_0 a_n \\ \vdots & \ddots & \vdots \\ a_n a_0 & \ldots & a_n^2 \end{pmatrix}$$

More generally, consider the map $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ defined by

$$\nu_d(a_0 : \ldots : a_n) = (a_0^d : a_0^{d-1} a_1 : \ldots : a_n^d)$$

(where on the left hand side all possible monomials of degree d should appear). Prove that ν_d is injective and that its image is a projective set defined by quadratic equations. Label the coordinates in $\mathbb{P}^{\binom{n+d}{d}-1}$ by $X_{i_1 \ldots i_d}$ $(0 \leq i_1 \leq \ldots \leq i_d \leq n)$ and identify a point of coordinates $\{a_{i_1 \ldots i_d}\}$ with the polynomial equation $\sum_{i_1 \ldots i_d} \frac{d!}{i_1! \ldots i_d!} a_{i_1 \ldots i_d} X_{i_1} \ldots X_{i_d}$ (assuming that char$\mathbb{K}$ is not positive and smaller than or equal to d). Prove that then the image of ν_d is identified with the set of homogeneous polynomials (up to a scalar) of degree d in the variables $X_0, \ldots, X_n$ that are a d-th power of a linear form.

Definition. The above map ν_d is called the *Veronese embedding* of $\mathbb{P}^n$ of degree d, and its image is called the *Veronese variety*. In the particular case $n = 1$, the Veronese variety is called *rational normal curve of degree d*.

Exercise 1.9. Prove that the rational normal curve of degree d in $\mathbb{P}^d$ (parametrized by $X_0 = t_0^d, X_1 = t_0^{d-1} t_1, \ldots, X_d = t_1^d$) is defined by the vanishing of the minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \ldots & X_{d-1} \\ X_1 & X_2 & \ldots & X_d \end{pmatrix}$$

Example 1.10. As we already said, the ideal of a rational normal curve is generated by the equations given in the above exercise. Let us prove it in the case $d = 3$ (the so-called *twisted cubic*). The idea for this kind of problems is to use division (a refinement of which is given by the so-called Gröbner basis, of which the interested reader can find a first introduction for instance in [CLO]). So let X be the image of the map $(t_0 : t_1) \mapsto (t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3)$ and take a homogeneous polynomial $F \in \mathbb{K}[X_0, X_1, X_2, X_3]$ that is in $I(X)$ (i.e. $F(t_0^3, t_0^2 t_1, t_0 t_1^2, t_1^3) = 0$ in $\mathbb{K}[t_0, t_1]$. Dividing F between $X_2^2 - X_1 X_3$ (which belongs to $I(X)$ and is monic in X_2) with respect to X_2 we get a relation

$$F = (X_2^2 - X_1 X_3) Q + X_2 A(X_0, X_1, X_3) + B(X_0, X_1, X_3)$$

with$Q$, A and B homogeneous polynomials. We look now for a monic polynomial in $I(X)$ not depending on X_2, and unfortunately we do not find any of degree two (which is the degree of what we know to be the generators), so we are constrained to use the auxiliary polynomial $X_1^3 - X_0^2X_3$ and we divide by it (with respect to X_1) the polynomials A and B, obtaining:

$$A = (X_1^3 - X_0^2X_3)G + X_1^2A_1(X_0, X_3) + X_1A_2(X_0, X_3) + A_3(X_0, X_3)$$

$$B = (X_1^3 - X_0^2X_3)H + X_1^2B_1(X_0, X_3) + X_1B_2(X_0, X_3) + B_3(X_0, X_3)$$

(and again the polynomials $G, H, A_1, A_2, A_3, B_1, B_2, B_3$ are homogeneous). Using now that $F(t_0^3, t_0^2t_1, t_0t_1^2, t_1^3) = 0$ and making substitutions in the above relations we get

$$t_0^5t_1^4A_1(t_0^3, t_1^3) + t_0^3t_1^3A_2(t_0^3, t_1^3) + t_0t_1^2A_3(t_0^3, t_1^3)+$$

$$+t_0^4t_1^2B_1(t_0^3, t_1^3) + t_0^2t_1B_2(t_0^3, t_1^3) + B_3(t_0^3, t_1^3) = 0$$

Collecting monomials according to their exponents modulo three, we obtain then equalities

$$t_0^5t_1^4A_1(t_0^3, t_1^3) + t_0^2t_1B_2(t_0^3, t_1^3) = 0$$

$$t_0^3t_1^3A_2(t_0^3, t_1^3) + B_3(t_0^3, t_1^3) = 0$$

$$t_0t_1^2A_3(t_0^3, t_1^3) + t_0^4t_1^2B_1(t_0^3, t_1^3) = 0$$

which respectively imply (here you need to use Proposition 1.1(i))

$$B_2(X_0, X_3) = -X_0X_3A_1(X_0, X_3)$$

$$B_3(X_0, X_3) = -X_0X_3A_2(X_0, X_3)$$

$$A_3(X_0, X_3) = -X_0B_1(X_0, X_3) = 0$$

A substitution of these relations in the expressions of F, A and B provides then

$$F = (X_2^2 - X_1X_3)Q + X_2(X_1^3 - X_0^2X_3)G + (X_1^3 - X_0^2X_3)H+$$

$$(X_1^2X_2 - X_0X_1X_3)A_1 + (X_1X_2 - X_0X_3)A_2 + (X_0X_2 - X_1^2)B_1$$

This proves that F is in the ideal generated by the polynomials $X_2^2-X_1X_3$, $X_2(X_1^3-X_0^2X_3)$, $X_1^2X_2 - X_0X_1X_3$, $X_1X_2 - X_0X_3$ and $X_0X_2 - X_1^2$ (which are in turn in $I(X)$). Since all of them are in the ideal generated by the minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix}$$

this proves that these minors generate $I(X)$, as announced.

Exercise 1.11. Prove that the ideal $I(X)$ canot be generated by just two homogeneous polynomials. Show that, however, the projective set defined by the ideal $I = (X_1X_3 - X_2^2, X_0^2X_3 - 2X_0X_1X_2 + X_1^3)$ is precisely the twisted cubic X. Prove the equality $\sqrt{I} = I(X)$.

Exercise 1.12. Prove that the homogeneous ideal of $V(X_0X_2 - X_1^2)$ is the ideal generated by $X_0X_2 - X_1^2$ (this fact will be immediate after Theorem 3.17).

Exercise 1.13. Prove that the set $X = \{(t_0^3 : t_0t_1^2 : t_1^3) \mid (t_0 : t_1) \in \mathbb{P}^1\}$ is a projective set of $\mathbb{P}^2$ and that $I(X)$ is the ideal generated by the polynomial $X_0X_2^2 - X_1^3$.

Exercise 1.14. Prove that $X = \{(t_0^4 : t_0^3t_1 : t_0t_1^3 : t_1^4) \in \mathbb{P}^3 \mid (t_0 : t_1) \in \mathbb{P}^1\}$ is a projective set and that $I(X) = (X_0X_3 - X_1X_2, X_1^3 - X_0^2X_2, X_2^3 - X_1X_3^2, X_1^2X_3 - X_0X_2^2)$

Example 1.15. As the reader was probably expecting, in this example we identify $\mathbb{P}^{\binom{n+1}{2}-1} = \mathbb{P}^{\frac{(n-1)(n+2)}{2}}$ with the set of skew symmetric $(n+1) \times (n+1)$ matrices modulo scalars, and consider the projective set of matrices of rank at most k. But this time the situation is quite different. First of all, a skew symmetric matrix has always even rank, so the projective sets of matrices of rank $2k+1$ will coincide with the set of matrices of rank $2k$. And on the other hand the ideal of this set is not generated now by the most "visible" equations. Consider for instance the case $n = 3$. Choose in $\mathbb{P}^5$ homogeneous coordinates $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$ to represent the matrix

$$\begin{pmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & p_{13} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & -p_{13} & -p_{23} & 0 \end{pmatrix}$$

Then it seems sensible to describe the set X of matrices of rank two by the vanishing of all the minors of order three. This provides homogeneous equations of degree three. However, as we remarked before, a skew symmetric matrix will have rank less than three if and only if it has rank less than four. And surprisingly enough, imposing rank less than four is just given by only one equation: the vanishing of the determinant. And the determinant looks a priori more complicated, since is homogeneous of degree four. But it is not so, the determinant of a skew symmetric matrix is always a perfect square. In our case, the determinant becomes $(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12})^2$, and hence X is just defined by the quadratic equation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$. For arbitrary n, the situation is that a skew-symmetric matrix has rank two if and only if all the principal minors of order four are zero. But as in the case $n = 4$, these minors, being again the determinant of a skew-symmetric matrix, are a perfect square of a quadratic form. Hence our projective set is defined by these quadratic forms, and in fact it is possible (but not trivial at all) to prove that these forms generate the homogeneous ideal.

Example 1.16. The above example is in fact a particular case of a more general and interesting construction, the *Grassmannian of k-planes in $\mathbb{P}^n$*. The idea is to give a structure of projective set to the set $\mathbb{G}(k,n)$ of all the linear spaces of dimension k in $\mathbb{P}^n$ (we will sometimes write $\mathbb{G}(k,M)$ to indicate the Grassmannian of k-planes in a projective space M, when we want to make M explicit). Of course $\mathbb{G}(k,n)$ is trivial when $k=0$ (the Grassmannian is then just $\mathbb{P}^n$) or $k=n-1$ (the Grassmannian being the dual projective space $\mathbb{P}^{n*}$). The key ingredient is the *Plücker embedding* $\mathbb{G}(k,n)\to\mathbb{P}^{\binom{n+1}{k+1}-1}$ that associates to the linear space Λ generated by $k+1$ linearly independent points $a_0,\dots,a_k\in\mathbb{P}^n$ the point in $\mathbb{P}^{\binom{n+1}{k+1}-1}$ whose coordinates are the maximal minors of the *Plücker matrix*

$$\begin{pmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix}$$

(where the rows are the coordinates of the $k+1$ points).

Exercise 1.17. Prove that the Plücker embedding is well defined (i.e. depends only on Λ but not on the choice of the points $a_0,\dots,a_k$ generating Λ) and that is injective.

We just need to see that the image of the Plücker embedding (which we will systematically identify with $\mathbb{G}(k,n)$) is a projective set. We first choose a good notation for the coordinates in $\mathbb{P}^{\binom{n+1}{k+1}-1}$ (called *Plücker coordinates*). We will denote with $p_{i_0\dots i_k}$ (with $0\le i_0<\dots<i_k\le n$) to the coordinate that would correspond to the determinant of the $(k+1)\times(k+1)$ matrix obtained by collecting the columns $i_0,\dots,i_k$ in the Plücker matrix.

Exercise 1.18. Prove that $\mathbb{G}(k,n)\cap V(p_{i_0\dots i_k})$ is the set of all k-planes meeting the linear space of equations $X_{i_0}=\dots=X_{i_k}=0$. Show with a counterexample that it is not true however that any hyperplane in $\mathbb{P}^{\binom{n+1}{k+1}-1}$ intersected with $\mathbb{G}(k,n)$ has such a nice geometrical description.

Let us study for instance the intersection of $\mathbb{G}(k,n)$ with $D(p_{0\dots k})$ (see Exercise 1.2). It is clear that a linear subspace Λ with $p_{0\dots k}\neq 0$ admits a Plücker matrix of the form

$$\begin{pmatrix} 1 & \dots & 0 & a_{0\,k+1} & \dots & a_{0n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & a_{k\,k+1} & \dots & a_{kn} \end{pmatrix}$$

Then the fact that a point of $\mathbb{P}^{\binom{n+1}{k+1}-1}$ belongs to $\mathbb{G}(k,n)\cap D(p_{0\dots k})$ is equivalent to say that we can multiply all the Plücker coordinates by a scalar (namely $\frac{1}{p_{0\dots k}}$) so that

$p_{0\dots k} = 1$, $p_{0\dots\hat{i}\dots k\,j} = (-1)^{k-i}a_{ij}$ (for $i = 0, \dots, k$ and $j = k+1, \dots, n$) and that the rest of the Plücker coordinates, being minors of the above Plücker matrix, are polynomial expressions in terms of the previous Plücker coordinates (to give a concrete example, for $k = 1, n = 3$ one finds $p_{01} = 1$, $p_{02} = a_{12}$, $p_{03} = a_{13}$, $p_{12} = -a_{02}$, $p_{13} = -a_{03}$ and $p_{23} = \begin{vmatrix} -p_{12} & -p_{13} \\ p_{02} & p_{03} \end{vmatrix} = -p_{12}p_{03} + p_{13}p_{02}$). Now the homogenization with respect to $p_{0\dots k}$ of these last equations produce homogeneous equations for $\mathbb{G}(k,n) \cap D(p_{0\dots k})$. Hence the product of these polynomials with $p_{0\dots k}$ vanish on the whole $\mathbb{G}(k,n)$. Collecting all these products when repeating the same procedure for all the Plücker coordinates we clearly obtain a set of polynomials defining $\mathbb{G}(k,n)$, so that the Grassmannian is a projective set.

Observe that, even if we have been lucky in the case $k = 1, n = 3$ obtaining precisely what we know to be the equation of $\mathbb{G}(k,n)$, in general we would obtain equations of degree $k+1$. This is still fine for $k = 1$, but not for bigger k, since it can be proved that the ideal of $\mathbb{G}(k,n)$ can be generated by quadratic polynomials. These quadratic polynomials can be obtained in the following way. Repeating the previous trick, for any $i_0 < \dots < i_k$ we get a particular Plücker matrix for the elements of $D(p_{i_0\dots i_k})$, and hence a particular way of getting $k+1$ points generating the corresponding k-plane. Hence, collecting again what we get for any coordinate, we get a bunch of points generating the k-plane. In the case $k = 1$ it is not difficult to see that these points are the rows of the matrix

$$\begin{pmatrix} 0 & p_{01} & \dots & p_{0n} \\ -p_{01} & 0 & \dots & p_{1n} \\ \vdots & & \ddots & \vdots \\ -p_{0n} & -p_{1n} & \dots & 0 \end{pmatrix}$$

Hence for an element of $\mathbb{G}(1,n)$ the rank of the above matrix must be two, and hence we recover the situation of Example 1.15.

Exercise 1.19. Verify that in the cases $n = 3$ and $n = 4$ we find the above matrices and that the corresponding quadratic equations define the respective Grassmannians of lines.

The above trick will produce however equations of degree $k+1$, so that the situation did not improve for $k > 1$. The idea now is to repeat the same trick "dually", i.e. to use the particular Plücker matrices found in each $D(p_{i_0\dots i_k})$ to describe our k-planes as intersection of $n-k$ independent hyperplanes. Collecting now these hyperplanes when varying the Plücker coordinate, we arrive to a dual description of our k-planes as the intersection of a bunch of hyperplanes (the word "dual" is not used arbitrarily: what we are in fact doing is to identify a k-plane in $\mathbb{P}^n$ as an $(n-k-1)$-plane in $\mathbb{P}^{n*}$ and taking its Plücker coordinates in the dual space). The condition that each of the points we found previously belong to each of the hyperplanes we found now gives us the wanted quadratic equations (called, as you can imagine, *Plücker equations*). To explain better how this works, let us turn again to the case $k = 1, n = 3$. In $D(p_{01})$ we had (from the particular Plücker matrix) any line generated by the points $(1 : 0 : -p_{12} : -p_{13})$ and $(0 : 1 : p_{02} : p_{03})$. This is the line obtained by intersecting the planes (once we homogenize with respect to p_{01}) $p_{12}X_0 - p_{02}X_1 + p_{01}X_2 = 0$ and $p_{13}X_0 - p_{03}X_1 + p_{01}X_3 = 0$. But now imposing the condition that the rows of the matrix

$$\begin{pmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & p_{13} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & -p_{13} & -p_{23} & 0 \end{pmatrix}$$

(which we know to generate the line with the given Plücker coordinates) are points of each of the two planes above provide either a tautological relation or the condition $p_{01}p_{13} - p_{02}p_{13} + p_{03}, p_{12} = 0$. The interested reader can find for instance in the chapter 6 of [H] an intrinsic interpretation of Grassmannians and their Plücker equations in terms of the skew-symmetric product of a vector space.

Exercise 1.20. Use the last method to obtain the Plücker equations of $\mathbb{G}(1,4)$, $\mathbb{G}(2,4)$ and $\mathbb{G}(2,5)$. Why are the equations of $\mathbb{G}(1,4)$ and $\mathbb{G}(2,4)$ so similar?

It is clear that the same projective set X can be described by many different sets of equations, the biggest one being $I(X)$, which is a radical ideal. However it is not always possible (and often neither convenient) to avoid non-radical ideals, since they arise in a natural way (this is in fact the starting point of the theory of schemes). Let us see some examples.

Example 1.21. Consider the conic $C = V(X_0X_2 - X_1^2) \subset \mathbb{P}^2$. This is also the rational normal curve of degree two, i.e. the set of points of the form $(t_0^2 : t_0t_1 : t_1^2)$. You can easily then see that the intersection with $L = V(X_2)$ consists of just the point $(1 : 0 : 0)$, which in some sense counts twice; for instance, substituting in the above parametrization we get $t_1^2 = 0$, which suggests that we get a double solution. In fact, L is tangent to C at $(1 : 0 : 0)$; even if we did not define yet this concept the reader could be already familiar with this notion for plane curves or just for projective conics (if not, it should be enough to dehomogenize with respect to X_0 to obtain the parabola $X_2 = X_1^2$, whose tangent line at $(0, 0)$ is obviously the line $X_2 = 0$). Looking at Proposition 1.1 it looks clear that, in the dictionary between projective sets and homogeneous ideals, intersections of sets correspond to sums of ideals. But $I(C) + I(L) = (X_0X_2 - X_1^2, X_2) = (X_1^2, X_2)$, which clearly is not a radical ideal. This is due to the fact that the intersection of the conic and the line consists of two "infinitely close points" (we will see in Example 3.6 that this expression is more than intuitive). In fact the ideal (X_1^2, X_2) keeps this infinitesimal information. If you take a curve $X = V(F) \subset \mathbb{P}^2$, the condition $I(X) \subset I(C \cap L) = (X_1, X_2)$ just means that X passes through the intersection point of C and L. However, the condition $I(X) \subset I(C) + I(L)$ means that "X passes through the intersection point in the direction of L. To see this, the above condition means that we can write F (we will assume for instance that F is irreducible) in the form $F = aX_0^{d-1}X_2+$ terms of bigger degree with respect to X_1, X_2. If $a \neq 0$, the reader with some knowledge of plane curves should recognize immediately that this means that the tangent line of X at $(1:0:0)$ is L (maybe dehomogenizing with respect to X_0 could also help). Hence the non-radical ideal contains much more information about how the intersection was obtained than the ideal of the intersection itself.

Example 1.22. It is not true however that non-radical ideals come always from tangential intersections. Consider for instance the projective set $X = \{(t_0^4 : t_0^3t_1 : t_0t_1^3 : t_1^4) \in \mathbb{P}^3 \mid (t_0 : t_1) \in \mathbb{P}^1\}$ of Exercise 1.14. If we intersect with the plane $\Pi = V(X_1 - X_2)$ we obtain four points corresponding to the four simple solutions of the equation $t_0^3t_1 = t_0t_1^3$. Hence we do not get any tangency in this intersection. However the ideal $I(X) + (X_1 - X_2) = (X_1 - X_2, X_0X_3 - X_2^2, X_2^3 - X_0^2X_2, X_2^3 - X_2X_3^2, X_2^2X_3 - X_0X_2^2)$ is not radical (for instance $X_2(X_0 - X_3)$ does not belong to it, but its square does). Hence this sum is not the ideal of the four points (this can also be seen because in the plane $V(X_1 - X_2)$ the four points are the intersection of two conics, while in the above ideal we only find one conic on that plane). The actual reason of this anomalous behavior (which is characteristic only of the projective space) is very deep and far beyond the scope of these notes. What happens in this particular case is that we are missing the monomial $t_0^2t_1^2$, which geometrically corresponds to the fact that X lives "naturally" in $\mathbb{P}^4$ (as a rational normal curve of degree four) rather than in $\mathbb{P}^3$, where it is obtained by a linear projection of another set in $\mathbb{P}^4$. In general, we obtain the same anomalous behavior when something is missing in the graded ring of X (only if the "missing part" has degree one it is possible to give a clear geometric interpretation as above).

Example 1.23. We will now consider one line moving to meet another one. Specifically, fix the line $L = V(X_2, X_3)\mathbb{P}^3$ and take another one $L_t = V(X_1, X_3 - tX_0)$. Clearly, when $t \neq 0$ the two lines are skew and the ideal of their union is $I_t = I(L \cup L_t) = (X_2, X_3) \cap (X_1, X_3 - tX_0) = (X_1X_2, X_1X_3, X_2X_3 - tX_0X_2, X_3^2 - tX_0X_3)$. If instead $t = 0$, then the two lines meet at one point and $I(L \cup L_0) = (X_2, X_3) \cap (X_1, X_3) = (X_3, X_1X_2)$. However we obtain a different result if we make $t = 0$ in the last expression for I_t. Indeed $I_0 = (X_1X_2, X_1X_3, X_2X_3, X_3^2)$, which is non-radical. We will see later on that, when regarding $L \cup L_0$ as a limit of $L \cup L_t$ it is more natural to consider I_0 rather than its radical.

The projective Hilbert's Nullstellensatz (Theorem 3.17) will tell us that whenever we have a projective set defined by a non-radical ideal, the homogeneous ideal will be just the radical of the given ideal (as it can be checked in the previous examples). For the time being let us check this result for ideals defining the empty set (which in fact behave slightly different with respect to the general case, since the ideal of the empty set would be (1) rather than $\mathfrak{M}$).

Theorem 1.24 (Weak Hilbert's Nullstellensatz)**.** *Let I be a homogeneous ideal. Then $V(I) = \emptyset$ if and only if for all $i = 0, \dots, n$ there exists $d_i \in \mathbb{N}$ such that $X_i^{d_i} \in I$ (or in other words, $\sqrt{I} \supset \mathfrak{M}$, i.e. $\sqrt{I} = \mathfrak{M}$ or $\sqrt{I} = (1)$).*

Proof: It is obvious that, if $X_0^{d_0}, \dots, X_n^{d_n} \in I$ then $V(I) = \emptyset$. Hence we only have to prove that if there exists $i \in \{0, \dots, n\}$ such that $X_i^d \notin I$ for any d then $V(I) \neq \emptyset$. We will prove it by induction. If $n = 0$, it is clear that if $X_0^d \notin I$ for any d and I is homogeneous then $I = (0)$, so the statement follows immediately.

So assume $n > 0$. For the sake of simplicity (and without loss of generality) we can assume $i = 0$, i.e. $X_0^d \notin I$ for any d. Consider $I' = I \cap \mathbb{K}[X_0, \dots, X_{n-1}]$. It is easy to see that I' is a homogeneous ideal of $\mathbb{K}[X_0, \dots, X_{n-1}]$, and since $X_0^d \notin I'$ for any d, from induction hypothesis we have that there exists $(a_0, \dots, a_{n-1}) \in \mathbb{K}^n \setminus \{(0, \dots, 0)\}$ vanishing at all polynomials of I'. Consider now $J = \{F(a_0, \dots, a_{n-1}, X_n) \mid F \in I\}$. Then J is an ideal of $\mathbb{K}[X_n]$ (not necessarily homogeneous).

I claim that J is not the whole $\mathbb{K}[X_n]$. If it were so, there would exist $F \in I$ such that $F(a_0, \dots, a_{n-1}, X_n) = 1$. So we could write $F = A_0 + A_1 X_n + \dots A_d X_n^d$, with $A_i \in \mathbb{K}[X_0, \dots, X_{n-1}]$, $A_1(a_0, \dots, a_{n-1}) = \dots = A_d(a_0, \dots, a_{n-1}) = 0$ and $A_0(a_0, \dots, a_{n-1}) \neq 0$. On the other hand, we can always assume that $(0 : \dots : 0 : 1) \notin V(I)$, since otherwise there is nothing to prove. This means that we can find a homogeneous polynomial $G \in I$ which is monic in the variable X_n. We write now $G = B_0 + B_1 X_n + \dots + B_{e-1} X_n^{e-1} + X_n^e$ with $B_j \in \mathbb{K}[X_0, \dots, X_{n-1}]$.

Let $R \in \mathbb{K}[X_0, \dots, X_{n-1}]$ be the resultant of F and G with respect to the variable X_n. In other words,

$$R = \begin{vmatrix} A_0 & A_1 & \dots & A_d & 0 & 0 & \dots & 0 \\ 0 & A_0 & \dots & A_{d-1} & A_d & 0 & \dots & 0 \\ & & \ddots & & & & & \\ 0 & \dots & 0 & A_0 & A_1 & \dots & A_{d-1} & A_d \\ B_0 & B_1 & \dots & B_{e-1} & 1 & 0 & \dots & 0 \\ 0 & B_0 & \dots & B_{e-2} & B_{e-1} & 1 & 0\dots & 0 \\ & & \ddots & & & & \ddots & \\ 0 & \dots & 0 & B_0 & B_1 & \dots & B_{e-1} & 1 \end{vmatrix}$$

It is then well-known that $R \in I$ (in the above matrix, add to the first column the second one multiplied by X_n, plus the third one multiplied by X_n^2, and so on till the last one multiplied by X_n^{d+e-1}; developing the resulting matrix by the first column you will find that R is a linear combination of F and G). Therefore $R \in I'$. But a direct inspection at the above determinant defining the resultant shows that, when evaluating at $(a_0, \ldots, a_{n-1})$, it becomes the determinant of a lower-triangular matrix, whose entries at the main diagonal are all 1. Hence $R(a_0, \ldots, a_{n-1}) = 1$, which contradicts the fact that $R \in I'$. This proves the claim.

Therefore J is a proper ideal of $\mathbb{K}[X_n]$, and hence (exercise 0.5) it is generated by a polynomial $f(X_n)$ of positive degree (or f is zero). Since $\mathbb{K}$ is algebraically closed, f has at least one root $a_n \in \mathbb{K}$. This means that $(a_0 : \ldots : a_n) \in V(I)$, which completes the proof. □

Remark 1.25. Of course the above result is false if $\mathbb{K}$ is not algebraically closed. For instance, $(X_0^2 + X_1^2)$ defines the empty set in the real projective line, and however is a radical ideal.

2. Irreducible components

Definition. A subset $X \subset P$ of a topological space P is called *irreducible* if it satisfies any of the following (clearly equivalent) properties:

(i) X cannot be expressed as a union $X = Z_1 \cup Z_2$, with $Z_1 \subsetneq X$ and $Z_2 \subsetneq X$ closed subsets of X (with the induced topology).

(ii) If $X \subset Z_1 \cup Z_2$ (with Z_1 and Z_2 closed sets of P) then either $X \subset Z_1$ or $X \subset Z_2$.

(iii) Any two nonempty open sets of X necessarily meet.

If $P = \mathbb{P}^n$ with the Zariski topology and X is a projective (resp. quasiprojective) set, then X is called a *projective (resp. quasiprojective) variety.*

Even if this definition is done for arbitrary quasiprojective sets, the following two lemmas will show that it is enough to study irreducibility for projective sets, and that this is done by just inspecting the homogeneous ideal.

Lemma 2.1. *Let $X \subset \mathbb{P}^n$ be a quasiprojective set and let $\overline{X}$ denote its Zariski closure.*

(i) X is the intersection of $\overline{X}$ with some open set.

(ii) X is irreducible if and only if $\overline{X}$ is irreducible.

Proof: By definition, $X = Y \cap U$, where $Y \subset \mathbb{P}^n$ is a closed subset containing X (and hence $\overline{X} \subset Y$). From the chain of inclusions $X \subset \overline{X} \cap U \subset Y \cap U$ we immediately get (i).

Assume first that $\overline{X}$ is irreducible. If $X \subset Z_1 \cup Z_2$ with Z_1, Z_2 closed subsets of $\mathbb{P}^n$ then obviously $\overline{X} \subset Z_1 \cup Z_2$, so that from the irreducibility of $\overline{X}$ either $\overline{X} \subset Z_1$ or $\overline{X} \subset Z_2$. Intersecting with U we conclude that X is irreducible.

Assume now that X is irreducible. If $\overline{X} \subset Z_1 \cup Z_2$ with Z_1, Z_2 closed subsets of $\mathbb{P}^n$, then also $X \subset Z_1 \cup Z_2$, and hence either $X \subset Z_1$ or $X \subset Z_2$. Taking closures we conclude that $\overline{X}$ is irreducible, which completes the proof of the lemma. □

Lemma 2.2. *A projective set X is irreducible if and only if its homogeneous ideal $I(X)$ is prime.*

Proof: Assume first X is irreducible. Let F, G be homogeneous polynomials such that $FG \in I(X)$. Then clearly $X \subset V(F) \cup V(G)$, so that the irreducibility implies that either $X \subset V(F)$ or $X \subset V(G)$. But the latter is equivalent to $F \in I(X)$ or $G \in I(X)$, which proves that $I(X)$ is prime.

Assume now that $I(X)$ is prime and $X \subset Z_1 \cup Z_2$ with Z_1, Z_2 projective sets. Suppose $X \not\subset Z_1$ and $X \not\subset Z_2$. Then $I(Z_1) \not\subset I(X)$ and $I(Z_2) \not\subset I(X)$. We can therefore find homogeneous polynomials $F \in I(Z_1)$ and $G \in I(Z_2)$ none of them in $I(X)$. But $FG \in$

$I(Z_1 \cup Z_2) \subset I(X)$, which contradicts the fact that $I(X)$ is prime. This completes the proof of the Lemma. $\square$

Exercise 2.3. Prove that linear spaces and the Segre and Veronese varieties are irreducible sets.

We want to prove next that any projective set can be decomposed into irreducible sets. Even if this can be checked directly (in the same spirit that Lemma 2.4 below), we will derive it from existence of the primary decomposition, since we will need to use it later on.

The following two lemmas will immediately imply the existence of a primary decomposition for any homogeneous ideal.

Lemma 2.4. *Any homogeneous ideal of S can be expressed as a finite intersection of homogeneous irreducible ideals (a homogeneous ideal I will be called irreducible if I cannot be expressed as $I = I_1 \cap I_2$, with $I \subsetneq I_1$ and $I \subsetneq I_2$ homogeneous ideals).*

Proof: Assume there exists a homogeneous ideal I that is not a finite intersection of homogeneous irreducible ideals. In particular, I itself is not irreducible, so that it can be expressed as a non-trivial intersection of two homogeneous ideals I_1 and J_1. From our hypothesis, it is clear that both I_1 and J_1 cannot be a finite intersection of homogeneous irreducible ideals. Assume for instance that I_1 is not a finite intersection of homogeneous irreducible ideals. Putting I_1 instead of I in the previous reasoning, we will find I_2 strictly containing I_1 and such that I_2 is not a finite intersection of homogeneous irreducible ideals. Iterating the process we would get an infinite chain $I \subsetneq I_1 \subsetneq I_2 \subsetneq \ldots$, which contradicts the fact that S is noetherian. $\square$

Lemma 2.5. *Any homogeneous irreducible ideal is primary.*

Proof: Let I be a homogeneous irreducible ideal, and assume that we have two homogeneous elements $F, G \in S$ such that $FG \in I$. For each $n \in \mathbb{N}$ consider the ideal $I_n = \{H \in S \mid HF^n \in I\}$. Since we have a chain $I = I_0 \subset I_1 \subset I_2 \subset \ldots$ and S is noetherian, there exists an $n \in \mathbb{N}$ such that $I_n = I_{n+1}$. Now I claim that $I = ((F^n)+I) \cap J$, where $J = \{H \in S \mid FH \in I\}$ (observe that both ideals in the intersection are homogeneous). Assuming for a while that the claim is true, this would imply from the irreducibility of I that either $I = (F^n) + I$ (and hence $F^n \in I$) or $I = J$ (and hence $G \in I$, since by assumption $G \in J$). Therefore the statement will follow as soon as we will prove the claim.

Take then $H \in ((F^n) + I) \cap J$. Hence $FH \in I$, and also we can write $H = AF^n + B$, with $A \in S$ and $B \in I$. Multiplying by F this last equality we get $AF^{n+1} = FH - FB \in I$.

Thus $A \in I_{n+1} = I_n$, so that $AF^n \in I$, which implies $H \in I$. This proves the non-trivial inclusion of the claim, and hence the lemma. $\square$

Exercise 2.6. Prove that the ideal $I = (X_1^2, X_1X_2, X_2^2) \subset \mathbb{K}[X_0, X_1, X_2]$ is primary, but it is not irreducible, since it can be written as $I = (X_1^2, X_2) \cap (X_1, X_2^2)$ (if you want an interpretation as at the end of Example 1.21, this decomposition is saying that a curve which passes through the point $(1:0:0)$ in two different tangent directions is necessarily singular). Is this decomposition unique?

Theorem 2.7. *Any homogeneous ideal $I \subset S$ can be written as $I = I_1 \cap \ldots \cap I_s$, where each I_i is primary, the radicals $\sqrt{I_1}, \ldots, \sqrt{I_s}$ are all different and for each $i = 1, \ldots, s$ $I_i \not\supset \bigcap_{j \neq i} I_j$. Moreover, the primary ideals in the decomposition such that their radical is minimal among $\{\sqrt{I_1}, \ldots, \sqrt{I_s}\}$ are uniquely determined by I, in the sense that they appear in each decomposition as above.*

Proof: It is clear from lemmas 2.4 and 2.5 that I can be written as a finite intersection of primary ideals. The condition $I_i \not\supset \bigcap_{j \neq i} I_j$ can be easily obtained by just removing from the decomposition any primary ideal containing the intersection of the others. Also, from Exercise0.2(iv), the intersection of all the primary ideals with the same radical is also primary, so that we can also assume that the radical ideal of the primary components are all different. So we are only left to prove the uniqueness statement.

Assume thus that we have two decompositions $I = I_1 \cap \ldots \cap I_s = I'_1 \cap \ldots \cap I'_t$ as in the statement. Assume for instance that $\sqrt{I_l}$ is minimal among $\sqrt{I_1}, \ldots, \sqrt{I_s}$. Then $\sqrt{I_l} \not\supset \bigcap_{j \neq l} \sqrt{I_j}$, by Exercise 0.1(v). We can then find $F \in \bigcap_{j \neq l} \sqrt{I_j}$ not belonging to I_l. Thus since $F \notin \sqrt{I} = \bigcap_j \sqrt{I'_i}$ it follows that we can find some $i = 1, \ldots, t$ such that $F \notin \sqrt{I'_i}$, and clearly we can take $\sqrt{I'_i}$ to be minimal (since $\sqrt{I}$ is the intersection of the minimal ideals). Let us see that in this situation $I_l \subset I'_i$.

Indeed, take $G \in I_l$. Since there is a power F^d of F belonging to $\bigcap_{j \neq l} I_j$, then F^dG belongs to $I = I'_1 \cap \ldots \cap I'_t$. In particular $F^dG \in I'_i$. But $F^d \notin I'_i$ and therefore $G \in I'_i$. We thus get the inclusion $I_l \subset I'_i$, and in a similar way (since $\sqrt{I'_i}$ was minimal) we get that I'_i is contained in some I_j. But the irredundance of the decomposition implies then $I_l = I_j$ and hence $I_l = I'_i$, completing the proof of the theorem. $\square$

Definition. A primary decomposition as in the statement of Theorem 2.7 is called *irredundant.* The primary ideals corresponding to non-minimal prime ideals appearing in an irredundant decomposition are called *embedded components* of I. The radical ideals of the primary components of an irredundant decomposition are called *associated primes* of the ideal.

A much stronger result is true (see Theorem 4.10) and in a wider context, but we just proved here what we needed for our geometric purposes.

Exercise 2.8. Show that the ideal I_0 of Example 1.23 has as primary decomposition $I_0 = (X_1, X_3) \cap (X_2, X_3) \cap (X_1 + aX_3, X_2 + bX_3, X_3^2)$ for any choice of $a, b \in \mathbb{K}$, and hence the embedded primary component is not unique (the geometric interpretation is that the ideal still "remembers" that the intersection point of the lines came from outside the plane $V(X_3)$, and hence there is an embedded component at the point consisting of some a tangent direction not contained in the plane; it does not matter which direction we take, which explains why the embedded component is not unique).

The important consequence of Theorem 2.7 is the following result (which also explains geometrically the uniqueness statement).

Corollary 2.9. *Any projective set decomposes in a unique way as a finite union of irreducible projective sets $X = Z_1 \cup \ldots \cup Z_s$, with $Z_i \not\subset Z_j$ if $i \neq j$.*

Proof: Let $I(X) = I_1 \cap \ldots \cap I_s$ a primary decomposition of $I(X)$. Since $I(X)$ is a radical ideal, taking radicals in the above expression allows us to assume that $I_1, \ldots, I_s$ are radical and hence prime. Removing redundant ideals we can assume that $I_j \not\subset I_i$ if $i \neq j$. Writing $Z_i = V(I_i)$ for each $i = 1, \ldots, s$, we have the wanted decomposition. Indeed since $X = Z_1 \cup \ldots \cup Z_s$, we have $I(X) = I(Z_1) \cap \ldots \cap I(Z_s)$. Hence each I_i contains $I(Z_1) \cap \ldots \cap I(Z_s)$, and thus Exercise 0.1(v) implies that it contains some $I(Z_j)$, which in turn contains I_j. Therefore $i = j$ and $I(Z_i) = I_i$, so that Z_i is irreducible.

In order to prove it is unique, assume there is another irredundant decomposition $X = Z'_1 \cup \ldots \cup Z'_t$. Hence $I(X) = I(Z'_1) \cap \ldots \cap I(Z'_t)$ is another primary decomposition of $I(X)$ in which all the primary ideals are prime and minimal. The uniqueness statement in Theorem 2.7 gives then a contradiction. □

Definition. The sets $Z_1, \ldots, Z_r$ in the statement of the above corollary are called the *irreducible components* of X.

Exercise 2.10. Find the irreducible components of $V(X_0X_3 - X_1X_2, X_1X_3 - X_2^2)$ [Hint: Observe that the two equations belong to the homogeneous ideal of the twisted cubic].

In the primary decomposition of a homogeneous ideal, we can certainly have a primary component defining the empty set. By the weak Nullstellensatz (Theorem 1.24), such a component must be $\mathfrak{M}$-primary (and hence we will call it an *irrelevant component.* Notice that there is at most one irrelevant component. We will show in the next results that we can essentially "get rid" of irrelevant components.

Lemma 2.11. *Assume that an ideal I has an irrelevant component I' and write $I = I_0 \cap I'$, where I_0 is the intersection of the remaining components. Then I_0 is the set of all the polynomials $F \in S$ such that $X_i^a F \in I$ for some $a \in \mathbb{N}$ and any $i = 0, \dots, n$.*

Proof: Since $\sqrt{I'} = \mathfrak{M}$, there is $a \in \mathbb{N}$ such that $X_i^a \in I'$ for $i = 0, \dots, n$. Therefore, for any $F \in I_0$ and $i = 0, \dots, n$, we have that $X_i^a F$ is in $I' \cap I_0$, i.e. belongs to I. Reciprocally, if $X_i^a F$ belongs to I for $i = 0, \dots, n$, we have that it belongs to any non-irrelevant primary component of I. But a non-irrelevant component of I cannot contain powers of all the variables $X_0, \dots, X_n$. Therefore, F must belong to any non-irrelevant component of I, i.e. F must belong to I_0. □

Definition. The *saturation of a homogeneous ideal* I is defined as the set $\operatorname{sat} I$ of polynomials $F \in \mathbb{K}[X_0, \dots, X_n]$ such that $X_i^a F \subset I$ for some $a \in \mathbb{N}$. It can be viewed as the maximum homogeneous ideal with the same dehomogenization as I with respect to any variable. An ideal is called *saturated* if it coincides with its saturation.

Proposition 2.12. *Let $I \subset \mathbb{K}[X_0, \dots, X_n]$ be a homogeneous ideal.*

(i) There exists l_0 such that $I_l = (\operatorname{sat} I)_l$ for $l \geq l_0$.

(ii) A homogeneous ideal $I' \subset \mathbb{K}[X_0, \dots, X_n]$ has the same saturation as I if and only if $I_l = I'_l$ for $l >> 0$.

(iii) If F is a homogeneous polynomial, then $I + (F)$ and $(\operatorname{sat} I) + (F)$ have the same saturation.

Proof: Since $I \subset \operatorname{sat} I$, to prove (i) it is enough to show that any element of $\operatorname{sat} I$ of sufficiently high degree is also in I. For this, we fix first a set of generators $F_1, \dots, F_s$ of $\operatorname{sat} I$. Choosing the maximum exponent, we can assume that there exists a such that $X_i^a F_j$ belongs to I for any $i = 0, \dots, n$ and $j = 1, \dots, s$. It is then clear that $GF_j \in I$ for any homogeneous polynomial of degree at least $(n+1)a$. Therefore, $l_0 = (n+1)a + \max\{\deg F_1, \dots, \deg F_s\}$ satisfies the required condition for (i).

For (ii), we first observe that (i) implies that two ideals with the same saturation coincide for high degree. We thus need to prove the converse, which is also easy. Indeed, let F be a polynomial in the saturation of I. This means that $X_i^a F$ belongs to I for some a. But we can take a big enough so that $\deg(X_i^a)$ is bigger than the degree for which I and I' coincide. Hence $X_i^a F$ is also in i', and therefore F is also in the saturation of I'.

Part (iii) is also very easy. In fact, (i) implies that I and $\operatorname{sat} I$ coincide for big degree. Therefore, $I + (F)$ and $(\operatorname{sat} I) + (F)$ also coincide for big degree, and thus (ii) imply that they have the same saturation. □

Exercise 2.13. Show that the ideal $I(X)+(X_1-X_2)$ in Example 1.22 is not saturated and that its saturation is the homogeneous ideal of $X\cap V(X_1-X_2)$. If you have some energy left, you could wish to prove that the primary decomposition of $I(X)+(X_1-X_2)$ is

$$(X_0,X_1,X_2)\cap(X_1,X_2,X_3)\cap(X_0-X_1,X_1-X_2,X_2-X_3)\cap(X_0+X_1,X_1-X_2,X_2+X_3)\cap I_\lambda$$

where $I_\lambda=(X_1-X_2,X_3-\lambda X_2,X_2-\lambda X_0,X_0^3)$, for any $\lambda\neq 0,1,-1$.

Exercise 2.14. Show that, if $L_1=V(X_0,X_1)$ and $L_2=V(X_2,X_3)$, then $I(L_1\cup L_2)+(X_1-X_2)=(X_0,X_1,X_2)\cap(X_1,X_2,X_3)\cap(X_1-X_2,X_0-\lambda X_1,X_3-\mu X_1,X_1^2)$ for any $\lambda,\mu\in\mathbb{K}$. In particular, the ideal $I(L_1\cup L_2)+(X_1-X_2)$ is not saturated its saturation is the homogeneous ideal of $(L_1\cup L_2)\cap V(X_1-X_2)$.

We end this chapter with an easy result that will show very useful later on.

Proposition 2.15. *Let $X,X'\subset\mathbb{P}^n$ two projective sets such that there is a Zariski open set $U\subset\mathbb{P}^n$ satisfying $X\cap U=X'\cap U$. Then the set of irreducible components of X meeting U coincides with the set of irreducible components of X' meeting U.*

Proof: Let Z be the complement of U. Then obviously $X\cup Z=X'\cup Z$. Clearly, the irreducible decomposition of $X\cup Z$ consists of the irreducible components of Z and the irreducible components of X not contained in Z (i.e. meeting U), and the same holds for $X'\cup Z$. The uniqueness of the irreducible decomposition immediately completes the proof. □

3. Hilbert polynomial. Nullstellensatz

Let I be a homogeneous ideal of $S = \mathbb{K}[X_0, \ldots, X_n]$. Clearly, the graded part of degree l of S/I is a vector space over $\mathbb{K}$, and has finite dimension because it is a quotient of S_l. We can then make the following:

Definition. Let I be a homogeneous ideal. The *Hilbert function* of I will be the map $h_I : \mathbb{N} \to \mathbb{N}$ defined by $h_I(l) = \dim_{\mathbb{K}}(S/I)_l$. The Hilbert function h_X of a projective set $X \subset \mathbb{P}^n$ will be the Hilbert function of its homogeneous ideal $I(X)$.

Warning: The above definition for the Hilbert function is different from the standard one in the following sense. In general, the Hilbert function of a graded S-module M is defined to be the map $h_M : \mathbb{N} \to \mathbb{N}$ associating to any l the dimension of P_l, the graded part of degree l of M. With this definition, the symbol h_I will have a precise different meaning; and what we called Hilbert function of I is in fact the Hilbert function of S/I. But since we are never going to use the general definition, I preferred to use this incorrect notation, since it is simpler to write h_I rather than $h_{S/I}$.

One of the main properties of the Hilbert function is that it is additive for exact sequences, and hence is somehow compatible with the sum and intersection of ideals, in the sense of the following lemma.

Lemma 3.1. *Let I_1, I_2 be two homogeneous ideal of S. Then there exists an exact sequence*

$$0 \to S/(I_1 \cap I_2) \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0$$

and therefore $h_{I_1 \cap I_2}(l) = h_{I_1}(l) + h_{I_2}(l) - h_{I_1+I_2}(l)$ for all $l \in \mathbb{Z}$.

Proof: Just define $\varphi : S/(I_1 \cap I_2) \to S/I_1 \oplus S/I_2$ by $\varphi(F + I_1 \cap I_2) = (F + I_1, F + I_2)$ and $\psi : S/I_1 \oplus S/I_2 \to S/(I_1 + I_2)$ by $\psi(G + I_1, H + I_2) = (G - H + I_1 + I_2)$. □

The importance of the Hilbert function is that, for large values of the degree l, it is given by a polynomial. In the case of a projective set, the coefficients of this polynomial for $I(X)$ will encode much information of how a X is contained in the projective space (and surprisingly enough, it also contains intrinsic information of X). The next lemma shows that this polynomial we are looking for must be zero for the empty set.

Lemma 3.2. *$h_I(l) = 0$ for $l >> 0$ if and only if $V(I) = \emptyset$.*

Proof: It is just a consequence of the weak Hilbert's Nullstellensatz (Theorem 1.24). □

Lemma 3.3. *Let $I \subset$ be a homogeneous ideal. Write $I = I_0 \cap I'$, where I' is (assuming it exists) the $\mathfrak{M}$-primary component of I, and I_0 the intersection of the remaining components. Then $h_I(l) = h_{I_0}(l)$ for $l >> 0$.*

Proof: This is immediate from Lemma 2.11 and Proposition 2.12. It is also a direct consequence of Lemma 3.1 and the fact that both I' and $I_0 + I'$ are $\mathfrak{M}$-primary (because their radical ideals are $\mathfrak{M}$, a maximal ideal), which allows us to apply Lemma 3.2. □

We move now to what intuitively must be zero dimensional: finite sets of points. We will see in the next examples and results that in this case there exists the wanted polynomial, which is constant, and its value is the number of points.

Example 3.4. We consider the easiest case, when X is one point. We can assume, in a suitable system of coordinates, that $X = \{(1 : 0 : \ldots : 0)\}$. Then, as shown in Example 1 in Chapter 1, $I(X) = (X_1, \ldots, X_n)$ and $S(X) \cong \mathbb{K}[X_0]$ as graded rings. Therefore $h_X(l) = 1$ for any $l \in \mathbb{N}$.

Exercise 3.5. Prove that $X = \{(1 : 0 : 0 : \ldots : 0), (0 : 1 : 0 : \ldots : 0)\}$, has homogeneous ideal $I(X) = (X_0X_1, X_2, \ldots, X_n)$, and graded ring $S(X) \cong \mathbb{K}[X_0, X_1]/(X_0X_1)$, and Hilbert function $h_X(l) = 2$ for $l \geq 1$, while $h_X(0) = 1$. Conclude that the Hilbert function of two different points in $\mathbb{P}^2$ is always like this.

Example 3.6. We consider now the ideal $I = (X_1^2, X_2, \ldots, X_n)$ corresponding to "two infinitely close points" (see Example 1.21). Then $S/I \cong \mathbb{K}[X_0, X_1]/(X_1^2)$ as graded rings, and therefore again $h_I(l) = 2$ if $l \geq 1$ and $h_I(0) = 1$. This shows that we again would get the constant polynomial 2, and therefore this polynomial we are looking for can even count infinitely close points.

Exercise 3.7. Compute the Hilbert function of three and four points in $\mathbb{P}^2$ and prove that it is respectively constant 3 and 4 for $l \geq l_0$ for some l_0. Discuss how small this l_0 can be depending on the relative position of the points.

After all these examples confirming that our wanted result should be true, we will prove it.

Proposition 3.8. *Let X be the set of d points in $\mathbb{P}^n$. Then $h_X(l) = d$ if $l \geq d - 1$. Reciprocally, if X is a projective set such that $h_X(l) = d$ for large values of l, then X is a set of d points.*

Proof: Let $p_1 = (a_{10} : \ldots : a_{1n}), \ldots, p_d = (a_{d0} : \ldots : a_{dn})$ be the points. Fixing vectors representing them, we can then define, for each $l \in \mathbb{N}$ the evaluation map $\varphi_l : S_l \to \mathbb{K}^d$ associating to each homogeneous polynomial $F \in S_l$ of degree l the d-uple

$(F(a_{10},\ldots,a_{1n}),\ldots,F(a_{d0},\ldots,a_{dn}))$. Then since clearly $I(X)_l$ is the kernel of φ_l, we have that $S(X)_l \cong \mathrm{Im}\varphi_l$. Hence our statement is equivalent to prove that φ_l is surjective if $l \geq d-1$. It is clear that, for each $i = 1,\ldots,d$ and $j \neq i$ we can find a linear form $H_i \in \mathbb{K}[X_0,\ldots,X_n]$ vanishing on p_i but not on any other p_j. Then the product $F_i = \Pi_{j\neq i} H_j$ is a homogeneous form of degree $d-1$ vanishing at all the points of X except p_i. Fixing a homogeneous form G of degree $l-d+1$, we get that the images by φ_l of the elements $GF_1,\ldots,GF_d$ generate $\mathbb{K}^d$. This proves the surjectivity of φ_l for $l \geq d-1$ and hence the first part of the proposition.

Reciprocally, assume that h_X takes a constant value d for $l >> 0$. If X were not finite, we could find $Z \subset X$ consisting of $d+1$ points and hence there is a surjection $S(X) \to S(Z)$. But, for large l the dimension of $S(X)_l$ is d while the dimension of $S(Z)_l$ is $d+1$ (using the part we already proved), so that we have a contradiction. This proves that X consists of a finite number of points. Of course, this number must be d, by the first part of the proposition. □

Exercise 3.9. Prove that, in the above proposition, $h_X(d-2) \neq d$ if and only if the d points are in a line.

Exercise 3.10. Prove that the Segre and Veronese varieties have Hilbert functions given by polynomials.

Exercise 3.11. Prove that the Hilbert function of the projective set in Exercise 1.14 coincides for degree $l \geq 2$ with the Hilbert function of the rational normal curve of degree four in $\mathbb{P}^4$.

Our next goal will be to proof that, as suggested by all the previous examples, the Hilbert function of any homogeneous ideal is given, for large values of the degree l, by a polynomial. The idea will be to take successive sections with hyperplanes. But, as Example 1.22 (together with exercise 2.13) shows, we need to be careful because this method could produce undesirable $\mathfrak{M}$-primary components.

Lemma 3.12. *Let F be a homogeneous polynomial of degree d not contained in any associated prime of an ideal I. Then there multiplication by F induces a graded monomorphism $(S/I)(-d) \to S/I$ and hence there is a graded exact sequence $0 \to (S/I)(-d) \to S/I \to S/(I+(F)) \to 0$. In particular $h_{I+(F)}(l) = h_I(l) - h_I(l-d)$*

Proof: Let $I = Q_1 \cap \ldots \cap Q_r$ be the homogeneous primary decomposition of I and let $P_1,\ldots,P_r$ be the respective associated primes. Take F not belonging to $P_1 \cup \ldots \cup P_r$. To prove that the multiplication by F is injective we just need to show that, for any homogeneous polynomial G, if $FG \in I$ then $G \in I$. But this is immediate from our hypothesis. In fact $FG \in I$ is equivalent to say that FG belongs to all the Q_i's ($i = 1, \ldots, r$). But Q_i being P_i-primary and F not belonging to P_i, this is equivalent to $G \in Q_i$ for all $i = 1, \ldots, r$. This is again equivalent to say $G \in I$, as wanted. □

We are now ready to prove the polynomial behavior of the Hilbert function.

Theorem 3.13. *Let I be a homogeneous ideal of $S = \mathbb{K}[X_0, \ldots, X_n]$. Then there exists a polynomial $P_I \in \mathbb{Q}[T]$ such that $h_I(l) = P_I(l)$ for $l >> 0$.*

Proof: We use induction on n. Obviously, for $n = 0$ there is nothing to prove since either $I = 0$ and hence $P_I = 1$ or I is M-primary and then $P_I = 0$.

Assume now $n > 0$. If I is $\mathfrak{M}$-primary, there is nothing to prove since again we can take P_I to be the zero polynomial. So assume that I is not $\mathfrak{M}$-primary. Lemma 2.11 and Proposition 2.12allow us to assume that $\mathfrak{M}$ is not an associated prime of I. Hence we can take a linear polynomial H not belonging to any associated prime of I, and so by Lemma 3.12 we see that the Hilbert function of $J = I + (H)$ satisfies $h_J(l) = h_I(l) - h_I(l-1)$. By changing coordinates we can assume $H = X_n$. Therefore, $S/J \cong \mathbb{K}[X_0, \ldots, X_{n-1}]/J'$, where J' is the set of all polynomials obtained from those of J after making the substitution $X_n = 0$. Hence the Hilbert function of J as an ideal of S can be regarded as the Hilbert function of J' as an ideal of $\mathbb{K}[X_0, \ldots, X_{n-1}]$. Therefore by induction hypothesis, there exists polynomial $P_J[T] \in \mathbb{Q}[T]$ such that $h_J(l) = p_J(l)$ for $l >> 0$.

It is now easy to construct a polynomial $Q(T) \in \mathbb{Q}[T]$ such that $Q(T) - Q(T-1) = P_J(T)$. Indeed, if P_J has degree d, by observing that $\binom{T}{0}, \ldots, \binom{T}{d}$ form a basis of the set fo polynomial of $\mathbb{Q}[T]$ of degree at most d, we can write $P_J(T) = a_0\binom{T}{0} + \ldots + a_d\binom{T}{d}$, with $a_i \in \mathbb{Q}$ (although not needed to our purpose, it can be seen that the coefficients belong actually to $\mathbb{Z}$, since this property characterizes polynomials taking integral values over $\mathbb{Z}$). Then the polynomial $Q(T) = a_0\binom{T+1}{1} + \ldots + a_d\binom{T+1}{d+1}$ clearly satisfies the required condition.

We consider now the difference $c(l) = h_I(l) - Q(l)$. From our construction, it follows that $c(l) - c(l-1) = 0$ for $l >> 0$, which immediately implies that c takes a constant value, say c_0, for $l >> 0$. Therefore $h_I(l)$ coincides with the polynomial function $Q(l) + c_0$ for large values of l, which completes the proof. □

Remark 3.14. The above theorem is true in the more general situation of a finitely generated graded module M over $\mathbb{K}[X_0, \ldots, X_n]$, with $\mathbb{K}$ not necessarily algebraically closed. The proof can be done by just using the long exact sequence of graded modules

$$0 \to M' \to M(-1) \xrightarrow{\cdot X_n} M \to M'' \to 0$$

where M' and M'' (which are now finitely generated graded modules over $\mathbb{K}[X_0, \ldots, X_{n-1}]$ are defined respectively as kernel and cokernel. A simple induction argument completes the proof (the lazy reader can check the details in [Mu]). I however preferred the above proof since it provides a geometric interpretation of the Hilbert polynomial. Another proof using Hilbert's syzygy theorem will be given in Chapter 4.

Definition. The *Hilbert polynomial of a homogeneous ideal* I is the polynomial P_I whose existence was proved in Theorem 3.13. The *Hilbert polynomial of a projective set* X is the polynomial P_X of its homogeneous ideal $I(X)$.

We can now translate into the language of Hilbert polynomials some of the results of this section.

Proposition 3.15. *The Hilbert polynomial satisfies the following properties:*

(i) $P_I = 0$ *if and only if* $V(I) = \emptyset$

(ii) If $I = I_0 \cap I'$ *with* I_0 *the* $\mathfrak{M}$*-primary component of* I *and* I' *the intersection of the other components of* I*. Then* $P_I = P_{I'}$*.*

(iii) P_X *is constant (and different from zero) if and only if* X *is a finite number of points. Moreover, in this case* P_X *is the number of points in* X *(see also Remark 5.3).*

(iv) $P_{I_1 \cap I_2} = P_{I_1} + P_{I_2} - P_{I_1+I_2}$.

(v) If F *is a homogeneous polynomial of degree* d *not contained in any relevant associated prime of* I*, then* $P_{I+(F)}(l) = P_I(l) - P_I(l-d)$*.*

Proof: Statement (i) is Lemma 3.2, (ii) is Lemma 3.3, (iii) is Proposition 3.8, (iv) is Lemma 3.1 and (v) is Lemma 3.12. □

We prove now a lemma that will justify our definition of dimension in the next section, and that will also help for the proof of the projective Hilbert's Nullstellensatz.

Lemma 3.16. *Let* I *be a non* $\mathfrak{M}$*-primary homogeneous ideal and consider the projective set* $X = V(I)$*. Then* $\deg P_I$ *is the maximum integer* m *such that any linear subspace of* $\mathbb{P}^n$ *of codimension* m *meets* X*.*

Proof: Let m be the degree of the Hilbert polynomial of I. By the weak Nullstellensatz and Proposition 3.15(i), $X \neq \emptyset$ and $m \geq 0$. Take a linear subspace $A \subset \mathbb{P}^n$ of codimension $r \leq m$, i.e. defined by r linearly independent forms $H_1, \ldots, H_r$. Hence from the exact sequence

$$(S/I)(-1) \xrightarrow{\cdot H_1} S/I \to S/(I + (H_1)) \to 0$$

it clearly follows that $P_{I+(H_1)}(l) \geq P_I(l) - P_I(l-1)$ for $l >> 0$. Since $P_I(l) - P_I(l-1)$ is a polynomial of degree $m-1$ with positive leading coefficient (m times the one of P_I), it follows that $P_{I+(H_1)}$ has degree at least $m-1$. Iterating the process we find that the Hilbert polynomial of $I+(H_1,\ldots,H_r)$ has degree at least $m-r\geq 0$, i.e. it is not the zero polynomial. Again from Proposition 3.15(i) it follows that $V(I+(H_1,\ldots,H_r))$ (which is nothing but $X\cap A$) is not empty.

In order to complete the proof we just need to find a linear subspace of codimension $m+1$ not meeting X. To this purpose, we just need to use Proposition 3.15(v). Since I is not $\mathfrak{M}$-primary, we can always find a linear form H_1 not belonging to any relevant associated prime of I, and hence the Hilbert polynomial of $I+(H_1)$ is $P_I(l)-P_I(l-1)$, which has degree $m-1$. We can repeat the same procedure finding linear forms $H_1,\ldots,H_{m+1}$ such that eventually the Hilbert polynomial of $I+(H_1,\ldots,H_m)$ is zero. This means, by Proposition 3.15(i), that X does not meet $V(H_1,\ldots,H_{m+1})$, which is a linear space of codimension $m+1$. □

Theorem 3.17 (Projective Hilbert's Nullstellensatz). *Let I be a non $\mathfrak{M}$-primary homogeneous ideal of $S=\mathbb{K}[X_0,\ldots,X_n]$, with $\mathbb{K}$ an algebraically closed field. Then $I(V(I))=\sqrt{I}$.*

Proof: Taking radicals in a primary decomposition of I, we find that $\sqrt{I}=I_1\cap\ldots\cap I_r$, where $I_1,\ldots,I_r$ are prime ideals. Since $I(V(I))=I(V(I_1))\cap\ldots\cap I(V(I_r))$, it is clear that it is enough to prove the theorem for prime ideals (since they are radical ideals).

Assume then that I is prime and take $F\in I(V(I))$ a homogeneous polynomial. If F is not in I, then from Proposition 3.15(v) it follows that the Hilbert polynomial of $I+(F)$ has degree $m-1$, where m is the degree of the Hilbert polynomial of I. But then Lemma 3.16 gives two contradictory results. On one hand, there exists a linear subspace of $\mathbb{P}^n$ of codimension m not meeting $V(I+(F))$. And on the other hand, all linear subspaces of $\mathbb{P}^n$ of codimension m should meet $V(I)$. Since $F\in I(V(I))$, then $V(I+(F))=V(I)$ and thus we have a contradiction. □

Corollary 3.18. *Let $X=V(F)$ and let $F=F_1^{a_1}\ldots F_s^{a_s}$ be the decomposition of F into irreducible factors (with $F_i\neq F_j$ if $i\neq j$). Then $I(X)=V(F')$, where $F'=F_1\ldots F_s$. In particular, $I(X)=(F)$ iff F is square-free.*

Proof: It is an immediate consequence of the Nullstellensatz. □

Exercise 3.19. Show that the Nullstellensatz is not true if the ground field is not algebraically closed.

I just want to finish this chapter by remarking that the affine Nullstellensatz (which is the standard one) can be obtained easily from the projective Nullstellensatz (see Theorems 14.2 and 14.4).

4. Graded modules; resolutions and primary decomposition

In this chapter we present several important results about graded modules, some of them generalizing what we have seen so far for ideals (for example the existence of the Hilbert polynomial and the primary decomposition). Most of these results will be used only at the end of these notes, when studying schemes. Therefore the reader interested just in projective varieties can skip the whole chapter, although the part of resolutions (which we will treat first) can be very instructive in a first approach to projective varieties.

Definition. A (finitely generated) *free graded module* over a graded ring S is a module M generated by a finite set of homogeneous elements that are linearly independent over S (called a *basis* of M).

Lemma 4.1. *Let M be a graded module over $S = \mathbb{K}[X_0, \dots, X_n]$.*

(i) M is free if and only if M is isomorphic to a module of the form $\oplus_i S(-a_i)$, with $a_i \in \mathbb{N}$.

(ii) If S is noetherian and M is finitely generated, then any submodule of M is also finitely generated.

Proof: If M is free and $m_1, \dots, m_r$ is a basis, set $a_i = \deg m_i$ for each $i = 1, \dots, r$. We thus have an isomorphism $\varphi : \oplus_i S(-a_i) \to M$ defined by $\varphi(F_1, \dots, F_r) = F_1 m_1 + \dots F_r m_r$. Reciprocally, if there is a graded isomorphism $\varphi : \oplus_i S(-a_i) \to M$, then the image of $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ is a basis of M. This proves (i)

In order to proof (ii), let $m_1 \dots, m_s$ be a system of graded generators of M, and let N be any submodule of M. We consider the set I of elements $F_s \in S$ such that there exists an element of the form $F_1 m_1 + \dots + F_s m_s$ in the submodule N. Clearly, I is an ideal of S, and therefore it is finitely generated. Let $n_1, \dots, n_r$ a set of elements of N such that, if we write $n_j = F_{j1} m_1 + \dots + F_{js} m_s$, then $F_{1s}, \dots, F_{rs}$ generate I. For any $n \in N$, we write it as $n = F_1 m_1 + \dots + F_s m_s$, and we thus get (since $F_s \in I$) that we can write $F_s = G_1 F_{1s} + \dots + G_r F_{rs}$ for some $G_1, \dots, G_r \in S$. This means that $n - G_1 n_1 - \dots - G_s n_r$ is in the submodule N' generated by $m_1, \dots, m_{s-1}$. We can repeat now the same trick for N' instead of N. It is clear that, iterating the process (or using induction on s), we get a finite set of generators for N. □

Definition. A *zerodivisor of a module M* over a ring S is an element $F \in S$ for which there exists a non-zero element $m \in M$ such that $Fm = 0$. A *torsion-free module* is a module whose only zerodivisor is 0.

Lemma 4.2. *Let* $0 \to M \xrightarrow{i} P \xrightarrow{f} N \to 0$ *be an exact sequence of graded* S*-modules. Then for any homogeneous* $F \in S$ *that is not a zerodivisor of* N *the induced sequence* $0 \to M/FM \xrightarrow{\bar{i}} P/FP \xrightarrow{\bar{f}} N/FN \to 0$ *is also exact.*

Proof: Clearly $\overline{f}$ is surjective and $\operatorname{Im}\bar{i} \subset \ker \overline{f}$. Reciprocally, if the class $\overline{p}$ of $p \in P$ modulo FP is in $\ker \overline{f}$, then $f(p) = Fn$ for some $n \in N$. Since f is surjective, there exists $p' \in P$ such that $f(p') = n$. Therefore $p - Fp' \in \ker f$, and hence there exists $m \in M$ such that $p = i(m) + Pp'$. This means that $\overline{p} = \bar{i}(\overline{m})$. We thus have $\operatorname{Im}\bar{i} = \ker \overline{f}$.

It remains to prove that $\bar{i}$ is injective. So assume $\bar{i}(\overline{m}) = 0$ for some $m \in M$. This means that $i(m) = Fp$ for some $p \in P$. We now have $0 = fi(m) = Ff(p)$. Since F is not a zerodivisor of N, then $f(p) = 0$, and therefore there exists $m' \in M$ such that $p = i(m')$. Hence $i(m) = i(Fm')$, and the injectivity of i implies $m = Fm'$, so that $\overline{m} = 0$. This completes the proof. $\square$

Theorem 4.3 (Hilbert's syzygy theorem)**.** *For any finitely generated graded module* M *over* $\mathbb{K}[X_0, \ldots, X_n]$, *there exists an exact sequence* $0 \to P_r \to \ldots \to P_0 \to M \to 0$ *in which* $P_0, \ldots, P_r$ *are free modules. Moreover, it is possible to find such an exact sequence with* $r \leq n+1$.

Proof: Let $m_1, \ldots, m_r$ be a set of homogeneous generators of M, with respective degrees $a_1, \ldots, a_r$. We have thus a surjective map $P_0 := \oplus_{i=1}^r S(-a_i) \to M$ defined by $(F_1, \ldots, F_r) \mapsto F_1 m_1 + \ldots + F_r m_r$. Let M_0 be the kernel of that map. We can now repeat the procedure to M_0 (which is again a finitely generated graded module) to obtain a surjection $P_1 \to M_0$, where P_1 is a free S-module. Iterating this process n times we arrive to an exact sequence $P_n \to \ldots \to P_0 \to M \to 0$, with $P_0, \ldots, P_n$ free S-modules. If P_{n+1} is now the kernel of the left-hand side map, the theorem will follows if we prove the following

Claim: If $0 \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0$ is any exact sequence of finitely generated graded modules over $S = \mathbb{K}[X_0, \ldots, X_n]$ and $P_0, \ldots, P_n$ are free, then also P_{n+1} is free.

We will prove the claim by induction. In case $n = 0$, we first observe that P_1 is torsion-free since it is a submodule of P_0. Let us see that this is enough to see that in this case P_1 is also free. Indeed take $m_1, \ldots, m_r$ to be a minimal set of homogeneous generators of P_1, and we have to prove that they are also linearly independent over $\mathbb{K}[X_0]$. So assume that we have a non-trivial linear combination $F_1 m_1 + \ldots + F_r m_r = 0$ with $F_1, \ldots, F_r \in \mathbb{K}[X_0]$. Splitting that equality into its homogeneous components, we can assume that each F_i is homogeneous, i.e. $F_i = \lambda_i X_0^{a_i}$, with $\lambda_i \in \mathbb{K}$ and $a_i \in \mathbb{N}$. We can also assume, reordering the elements, that the non-zero coefficients are the first ones $F_1, \ldots, F_s$ and that a_1 is the minimum among $a_1, \ldots, a_s$. We can thus divide the linear relation by $X_0^{a_1}$ (because P_1 is torsion-free) and get $\lambda_1 m_1 + \lambda_2 X_0^{a_2 - a_1} m_2 + \ldots + \lambda_s X_0^{a_s - a_1} m_s = 0$, with $\lambda_1 \neq 0$. This implies that m_1 is a superfluous generator, which contradicts our assumption. Hence P_1 is free.

So assume we know our claim to be true for $n-1$. Denote by f_i the map from P_i to P_{i-1} and by M_{i-1} its image (so $M_i = \ker f_{i-1} \subset P_{i-1}$ for $i = 1, \ldots, n$). Applying Lemma 4.2 to the exact sequences

$$0 \to P_{n+1} \to P_n \to M_n \to 0$$

$$0 \to M_n \to P_{n-1} \to M_{n-1} \to 0$$
$$\vdots$$
$$0 \to M_2 \to P_1 \to M_1 \to 0$$

(observe that $M_i \subset P_{i-1}$ for $i = 1, \ldots, n$, so that they do not have zerodivisors) we get a long exact sequence

$$0 \to P_{n+1}/X_nP_{n+1} \to P_n/X_nP_n \to \ldots \to P_2/X_nP_2 \to P_1/X_nP_1.$$

Since $\mathbb{K}[X_0, \ldots, X_n]/(X_n) \cong \mathbb{K}[X_0, \ldots, X_{n-1}] =: S'$, any P_i/X_nP_i $(i = 1, \ldots, n)$ can be regarded as a free graded S'-module, so that by induction hypothesis we have that also P_{n+1}/X_nP_{n+1} is free. Our scope is now to show that also P_{n+1} is free. So take $m_1, \ldots, m_r \in P_{n+1}$ such that there classes modulo X_nP_{n+1} form a basis for P_{n+1}/X_nP_{n+1}. We will prove that $m_1, \ldots, m_r$ form a basis of P_{n+1}.

Let a bar denote classes modulo X_nP_{n+1} for elements of P_{n+1} and classes modulo (X_n) for polynomials of S. Assume first that we have a non-trivial linear combination $F_1m_1 + \ldots + F_rm_r = 0$ with $F_1, \ldots, F_r \in S$. Since P_{n+1} is torsion-free, we can assume that not all $F_1, \ldots, F_r$ are divisible by X_n (otherwise we can divide the linear relation by the maximum common power of X_n). We therefore get another non-trivial relation $\overline{F}_1\overline{m}_1 + \ldots + \overline{F}_r\overline{m}_r = 0$. Since $\overline{m}_1, \ldots, \overline{m}_r$ form a basis for P_{n+1}/X_nP_{n+1}, we get that $\overline{F}_1 = \ldots = \overline{F}_r = 0$, i.e. all the F_i's are divisible by X_n, contrary to our assumptions. We therefore proved that $m_1, \ldots, m_r$ are linearly independent.

If $m_1, \ldots, m_r$ were not a set of generators, we could find a homogeneous element $m \in P$ not depending on $m_1, \ldots, m_r$. We choose such m with minimum degree among those satisfying that property. Since P_{n+1}/X_nP_{n+1} is generated by $\overline{m}_1, \ldots, \overline{m}_r$, we get that we can find a relation $m = F_1m_1 + \ldots + F_rm_r + x_nm'$ with $F_1, \ldots, F_r \in S$ and m' in P. Since $\deg m' < \deg m$, we can write m' as a linear combination of $m_1, \ldots, m_r$, and hence the same holds for m. This is a contradiction, which proves the claim and hence the theorem. □

Definition. An exact sequence like the one of Theorem 4.3 will be called a *free resolution of the module M*. The integer r will be called the *length of the resolution.*

We obtain from this corollary the existence in general of the Hilbert polynomial, even if the ground field $\mathbb{K}$ is not algebraically closed.

Corollary 4.4. *For any graded module M over $\mathbb{K}[X_0,\ldots,X_n]$ there exists a polynomial $P_M \in \mathbb{Q}[T]$ such that $\dim_{\mathbb{K}} M_l = P_M(l)$ if l is big enough.*

Proof: The result is clear if M is free. Indeed, if $M \cong \oplus_i S(-a_i)$, then $\dim_{\mathbb{K}} M_l = \Sigma_i \dim_{\mathbb{K}} \mathbb{K}[X_0,\ldots,X_n]_{l-a_i} = \Sigma_i \binom{l-a_i+n}{n}$ (the latter equality being true only if $l \geq a_i - n$ for each i), and this is a polynomial in l with rational coefficients. If M is arbitrary, take $0 \to P_r \to \ldots \to P_0 \to M \to 0$ to be a free resolution of M. Then $\dim_{\mathbb{K}} M_l = \dim_{\mathbb{K}}(P_0)_l - \dim_{\mathbb{K}}(P_1)_l + \ldots + (-1)^r \dim_{\mathbb{K}}(P_r)_l$, and as we have seen each of the summands is given by a rational polynomial if l is big enough. □

Warning: Observe that this notation is not consistent with the one we gave for ideals. In fact, what we called P_I should be written $P_{S/I}$ (which is in fact different from P_I in the sense we just described). We preferred however to use the first notation because it is simpler, and it will not be easy to yield any confusion, since the Hilbert polynomial of I as an S- module is not quite used.

Observe that a map $\bigoplus_{j=1}^{s} S(-a_j) \to \bigoplus_{i=1}^{r} (-b_i)$ is given by an $r \times s$ matrix whose (i,j)-entry is a homogeneous polynomial of degree $a_j - b_i$. Therefore giving a free resolution is the same thing as giving a series of matrices $A_1,\ldots,A_r$ with the condition $\ker A_i = \operatorname{Im} A_{i+1}$ (in particular $A_i A_{i+1} = 0$).

Example 4.5. Let us find a free resolution of the coordinate ring of the twisted cubic $X = \{(t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3) \in \mathbb{P}^3 \mid (t_0 : t_1) \in \mathbb{P}^1\}$. We know (Example 1.10) that $I(X)$ is the ideal generated by $X_0X_2 - X_1^2$, $X_1X_3 - X_2^2$ and $X_0X_3 - X_1X_2$. This yields the first part of the resolution $S(-2)^3 \xrightarrow{f} S \to S(X) \to 0$, where $f(A,B,C) = A(X_0X_2 - X_1^2) + B(X_1X_3 - X_2^2) + C(X_0X_3 - X_1X_2)$. The key point is to find the kernel of f. We can find immediately two elements $(X_2, X_0, -X_1)$ and $(X_3, X_1, -X_2)$ in the kernel, since there are relations

$$\begin{aligned} X_1(X_0X_3 - X_1X_2) &= X_2(X_0X_2 - X_1^2) + X_0(X_1X_3 - X_2^2) \\ X_2(X_0X_3 - X_1X_2) &= X_3(X_0X_2 - X_1^2) + X_1(X_1X_3 - X_2^2). \end{aligned} \qquad (*)$$

We want to show that these elements generate the whole kernel. So assume there is a relation

$$A(X_0X_2 - X_1^2) + B(X_1X_3 - X_2^2) + C(X_0X_3 - X_1X_2) = 0.$$

This implies in particular that $C(X_0X_3 - X_1X_2)$ belongs to the prime ideal (X_1, X_2). Since $X_0X_3 - X_1X_2 \notin (X_1, X_2)$, it follows that there exist polynomials D, E such that $C = DX_1 + EX_2$. Making this substitution in the above relation and using (*) we get

$$(A + DX_2 + EX_3)(X_0X_2 - X_1^2) + (B + DX_0 + EX_1)(X_1X_3 - X_2^2) = 0.$$

This implies the existence of a polynomial F such that

$$A + DX_2 + EX_3 = F(X_1X_3 - X_2^2)$$

$$B + DX_0 + EX_1 = -F(X_0X_2 - X_1^2)$$

We thus get

$$A = (-D - FX_2)X_2 + (-E + FX_1)X_3$$

$$B = (-D - FX_2)X_0 + (-E + FX_1)X_1$$

Since on the other hand we can write $C = DX_1 + EX_2 = (D + FX_2)X_1 + (E - FX_1)X_2$, we conclude that (A, B, C) is in the image of the map $g : S(-3)^2 \to S(-2)^3$ defined by $g(P, Q) = (PX_2 + QX_3, PX_0 + QX_1, -PX_1 - QX_2)$. But this map is injective, since $PX_2 + QX_3 = 0$ and $PX_0 + QX_1 = 0$ easily implies $(X_1X_2 - X_0X_3)P = 0$ and $(X_0X_3 - X_1X_2)Q = 0$, and hence $P = Q = 0$. Since it is also obvious that $\operatorname{Im} g \subset \ker f$ it follows that we have a resolution

$$0 \to S(-3)^2 \to S(-2)^3 \to S \to S(X) \to 0.$$

As a consequence, $\dim_{\mathbb{K}} S(X)_l = \dim_{\mathbb{K}} S_l - 3\dim_k S_{l-2} + 2\dim_{\mathbb{K}} S_{l-3} = \binom{l+3}{3} - 3\binom{l+1}{3} + 3\binom{l}{3} = 3l + 1$, and the equality works for any $l \geq 0$, as we already knew. Note that the matrices here are

$$A_1 = \begin{pmatrix} X_0X_2 - X_1^2 & X_1X_3 - X_2^2 & X_0X_3 - X_1X_2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} X_2 & X_3 \\ X_0 & X_1 \\ -X_1 & -X_2 \end{pmatrix}$$

Observe that the entries of A_1 are, up to a sign, the minors of A_2.

Exercise 4.6. Find a free resolution for a set of two, three or four points in $\mathbb{P}^2$, and reobtain that the Hilbert polynomial is the number of points (of course, the type of resolution will depend on the relative position of the points).

Exercise 4.7. Prove that the graded ring of the set X of Exercise 1.14 has a free resolution of the type $0 \to S(-5) \to S(-4)^4 \to S(-2) \oplus S(-3)^3 \to S \to S(X) \to 0$ (you will need to be patient). Check its Hilbert polynomial with Exercise 3.11.

Exercise 4.8. Prove that the graded ring of the disjoint union of two lines in $\mathbb{P}^3$ has a resolution of the type $0 \to S(-4) \to S(-3)^4 \to S(-2)^4 \to S \to S(X) \to 0$, while the graded ring of a line has a resolution $0 \to S(-2) \to S(-1)^2 \to S \to S(L) \to 0$.

We finish this section by generalizing to modules the primary decomposition given in Chapter 2 for ideals. From now on S will be an arbitrary graded ring, and in fact our proof will also work in the non homogeneous case.

Definition. A *primary graded submodule* of a graded S-module M is a graded submodule N such that for any $F \in S$ and $m \in M$ such that $Fm \in N$ then either $m \in N$ or there exists $l \in \mathbb{N}$ such that $F^l M \subset N$ (i.e. for any $m' \in M$ then $F^l m \in N$). The set $P := \{F \in S \mid F^l M \subset N \text{ for some } l \in \mathbb{N}\}$ is thus a homogeneous prime ideal, and N is said to be P*-primary.*

Theorem 4.9. *Let M be a noetherian graded S-module. Then any submodule N of M admits a decomposition $N = N_1 \cap \ldots \cap N_r$ such that each N_i is P_i-primary (with $P_i \neq P_j$ if $i \neq j$) and $\bigcap_{j \neq i} N_j \not\subset N_i$.*

Proof: It follows the same steps as for ideals. First using the noetherianity of M (since it is finitely generated over a noetherian ring) we observe, as in Lemma 2.4 that N is a finite intersection of irreducible modules (irreducible meaning not to be expressible as a non-trivial intersection of two modules).

Next we imitate the proof of Lemma 2.5 to conclude that any irreducible module N is primary. Indeed if we have $F \in S$ and $m \in M$ such that $Fm \in N$, then we can write $N = (F^l M + N) \cap \{m' \in M \mid Fm' \in N\}$, where l satisfies that $\{m' \in M \mid F^l m' \in N\} = \{m' \in M \mid F^{l+1} m' \in N\}$, and the proof is the same as in Lemma 2.5.

Finally, removing redundant components and gathering primary components with the same prime, we eventually get the wanted decomposition. □

Definition. A decomposition as in the statement of Theorem 4.9 will be called an *irredundant primary decomposition* of N.

We prove now a uniqueness theorem for irredundant primary decompositions. Exercise 2.8 shows that it is the best possible result.

Theorem 4.10. *Let $N = N_1 \cap \ldots \cap N_r$ be an irredundant primary decomposition in which each N_i is P_i-primary. Then*

(i) For each homogeneous $m \in M$, the set $I_m := \{F \in S \mid F^l m \in N \text{ for some } l \in \mathbb{N}\}$ is a homogeneous ideal and $I_m = \bigcap_{m \notin N_i} P_i$.

(ii) The set $\{P_1, \ldots, P_r\}$ coincide with the set of ideal I_m that are prime. In particular, $\{P_1, \ldots, P_r\}$ does not depend on the primary decomposition.

(iii) For each homogeneous $F \in S$, the set $M_F := \{m \in M \mid F^l m \in N \text{ for some } l \in \mathbb{N}\}$ is a graded submodule of M and $M_F = \bigcap_{F \notin P_i} N_i$.

(iv) *If P_i is minimal in the set $\{P_1,\dots,P_r\}$, then N_i does not depend on the primary decomposition.*

Proof: To prove (i) it is enough to prove the equality $I_m=\bigcap_{m\notin N_i}P_i$. So let us prove the double inclusion. The first inclusion is clear: if we have $F\in I_m$, then some F^lm is in N, hence in in any N_i, and if $m\notin N_i$ it follows from the primality of N_i that F belongs to P_i. Reciprocally, if F belongs to $\bigcap_{m\notin N_i}P_i$, then for each i such that $m\notin \mathbb{N}_i$ we have $F\in P_i$, so that there exists some $F^{l_i}m$ in N_i. Taking l to be the maximum of these l_i's we obtain that F^lm is in any N_i not containing m. Since obviously F^lm is in any N_i containing m, we get $F^lm\in N_1\cap\dots\cap N_r=N$. Therefore F is in I_m, which proves (i).

To prove (ii), we first observe that if some I_m is prime, then it should coincide with some P_i, by using Exercise 0.1(v) and the fact we just proved that I_m is a finite intersection of P_i's. On the other hand, since the primary decomposition is irredundant, we can find for each $i=1\dots,r$ an element $m_i\in\bigcap_{j\neq i}N_j\setminus N_i$, and therefore (i) implies $I_{m_i}=P_i$.

The prove of (iii) is completely analogous to the proof of (i). Finally, the proof of (iv) is like the end of the proof of (ii), with the difference that only if P_i is minimal in the set $\{P_1,\dots,P_r\}$ it is possible to find $F_i\in\bigcap_{j\neq i}P_j\setminus P_i$ (again Exercise 0.1(v) shows that $\bigcap_{j\neq i}P_j\subset P_i$ if and only if P_i is contained in some P_j with $j\neq i$). We thus have that N_i coincides with M_{F_i}, and therefore it does not depend on the primary decomposition. □

Definition. The prime ideals $P_1,\dots,P_r$ of the above theorem are called the *associated primes of the submodule N*. The primary components corresponding to non-minimal prime ideals are called *embedded components of the submodule N*.

Remark 4.11. From the above proof one could think that the primary decomposition of N is related to the primary decomposition (in the sense of Chapter 2) of the ideal $\{F\in S\mid FM\subset N\}$ and more precisely that the primary components of the latter are $\{F\in S\mid FM\in N_i\}$. However this is not true. Take for instance $M=S/I\oplus S/J$, with $I\subset J$ two prime ideals of S. Then $(0)=\big(S/I\oplus(0)\big)\cap\big(S/J\oplus(0)\big)$ is the primary decomposition of (0), while $\{F\in S\mid FM=0\}=I$.

Definition. *A zerodivisor of a module M* is an element $F\in S$ for which there exists $m\in M\setminus\{0\}$ such that $Fm=0$.

Proposition 4.12. *Let M be a noetherian graded module and let $P_1\dots P_r$ be the associated primes of a submodule N. Then $P_1\cup\dots\cup P_r$ is the set of zerodivisors of M/N, i.e. the elements $F\in S$ for which there exists $m\in M\setminus N$ such that $Fm\in N$. In particular, the set of zerodivisors of M is the union of the associated primes of (0).*

Proof: Let first F be a zerodivisor of M/N. Thus there exists an element $m \in M \setminus N$ such that $Fm \in N$. Since $m \notin N$, then there exists $i = 1, \ldots, r$ such that $m \notin N_i$. But on the other hand Fm is in N, so that it belongs to N_i. Now the fact that N_i is P_i-primary implies that F belongs to P_i.

Reciprocally, assume that we have $F \in P_i$ for some $i = 1, \ldots, r$. By Theorem 4.10(ii) there exists some $m \in M$ such that $P_i = I_m$. Therefore there exists $l \in \mathbb{N}$ such that $F^l m \in N$, and we take l to be minimum satisfying this condition (l must be at least 1 because $m \notin N$ since otherwise $1 \in I_m$). Thus $F^{l-1}m \notin N$ and $F(F^{l-1}m) = F^l m \in N$, which means that F is a zerodivisor of M/N. □

Definition. The *support of a module* M is the projective set $V(P_1) \cup \ldots \cup V(P_r)$, where $P_1, \ldots, P_r$ are the associated primes of (0).

5. Dimension, degree and arithmetic genus

We introduce in this chapter the main invariants of a projective set, which will be obtained form its Hilbert polynomial. We start with a very simple but useful remark (left as an exercise), which will be implicitly used throughout these notes. It just says that the invariants of a projective set $X \subset \mathbb{P}^n$ contained in some linear subspace H do not depend on whether we regard X as a projective set in $\mathbb{P}^n$ or H.

Exercise 5.1. Let $X \subset \mathbb{P}^n$ be a projective set. Prove that the Hilbert polynomial (and in fact the Hilbert function) of X is the same if X is considered as a projective set in $\mathbb{P}^m$ (with $m > n$) when regarding $\mathbb{P}^n$ as a linear subspace of $\mathbb{P}^m$.

The first main invariant, whose definition the reader should be hopefully guessing after the previous chapter, is the dimension.

Definition. Let $X \subset \mathbb{P}^n$ be a projective set. The *dimension* of X will be the degree of the Hilbert polynomial P_X. More generally, we can speak of the *dimension of an ideal* I as the degree of its Hilbert polynomial. A projective set of dimension one is called a *curve*, while a set of dimension two is called a *surface*.

The following result states that we can in fact define the dimension by using the Hilbert polynomial of any homogeneous ideal defining our projective set.

Lemma 5.2. *Let I be a homogeneous ideal of $\mathbb{K}[X_0, \dots, X_n]$ and let $X = V(I) \subset \mathbb{P}^n$ be the projective set it defines. Then $\dim X$ is the degree of P_I.*

Proof: From Lemma 3.16 applied to $I(X)$, the dimension of X is the maximum integer m such that any linear subspace of $\mathbb{P}^n$ of codimension m meets X. But using again Lemma 3.16, now applied to I, we obtain that this integer coincides with the degree of P_I. □

Remark 5.3. Observe that this lemma allows to strengthen Proposition 3.15(iii), in the sense that an ideal I has constant Hilbert polynomial if and only if $V(I)$ is a finite number of points. Indeed, since P_I and $P_{V(I)}$ have the same degree, P_I is constant if and only if $P_{V(I)}$ is, and the latter is constant if and only if $V(I)$ is a finite number of points.

Exercise 5.4. Prove that (fortunately!) this definition of dimension coincides for linear spaces with the usual definition of dimension of a linear subspace.

Exercise 5.5. Prove that the Veronese variety (image of $\mathbb{P}^n$) has dimension n, and that the Segre variety (image of $\varphi_{n,m}$) has dimension $n + m$. In particular, a rational normal curve has dimension 1.

Exercise 5.6. Show that, if $F \in \mathbb{K}[X_0, \dots, X_n]$ is a homogeneous polynomial of positive degree, then $V(F)$ has dimension $n - 1$.

Proposition 5.7. *The dimension of projective sets satisfies the following properties:*

(i) If $X \subset Y$, then $\dim X \leq \dim Y$.

(ii) Let X be a projective set of dimension r and F a homogeneous polynomial not containing any irreducible component of X. Then $X \cap V(F)$ has dimension $r-1$.

(iii) If $X \subset Y$, $\dim X = \dim Y$ and Y is irreducible, then $X = Y$.

(iv) The dimension of X is the maximum of the dimensions of the irreducible components of X.

(v) The dimension of a projective set X is the maximum length r of a strictly increasing chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_r$ of irreducible closed sets contained in X.

(vi) The dimension of a projective set $X \subset \mathbb{P}^n$ is the maximum integer m such that any linear subspace of $\mathbb{P}^n$ of codimension m meets X.

(vii) The dimension of a projective set X is zero if and only if X is a finite number of points.

(viii) If X has dimension r, then for any set of homogeneous polynomials $F_1, \ldots, F_s$ (with $s \leq r+1$), the projective set $X \cap V(F_1, \ldots, F_s)$ has dimension at least $r-s$ (and hence it is not empty if $s \leq r$).

Proof: Part (i) is just a consequence of the natural surjection $S(Y) \to S(X)$. For (ii), we first obtain from Lemma 3.12 that the degree of the Hilbert polynomial of $I + (F)$ is $r-1$. But since $V(I+(F)) = X \cap V(F)$, it follows from Lemma 5.2 that the degree of the Hilbert polynomial of $I + (F)$ is the dimension of $X \cap V(F)$, thus proving (ii).

In order to prove (iii), assume $X \subsetneq Y$. Then $I(Y) \subsetneq I(X)$, so that we can find a homogeneous polynomial $F \in I(X) \setminus I(Y)$. Hence $X \subset Y \cap V(F)$. From (i) and (ii) we conclude that then $\dim X \leq \dim(Y \cap V(F)) = \dim Y - 1$, which is a contradiction with the assumption $\dim X = \dim Y$.

As for (iv), let $X = X_1 \cup \ldots \cup X_s$ be the decomposition of X into irreducible components. We prove the assertion by induction on s, the case $s = 1$ being trivial. From (i) we know that $\dim X_i \leq \dim X$ for all i, so we just need to find a component with the same dimension as X. Writing $I(X) = I(X_1 \cup \ldots \cup X_{s-1}) \cap I(X_s)$ we can apply Lemma 3.1 to conclude that $P_X = P_{X_1 \cup \ldots \cup X_{s-1}} + P_{X_s} - P_{I(X_1 \cup \ldots \cup X_{s-1}) + I(X_s)}$. But observe that $V(I(X_1 \cup \ldots \cup X_{s-1}) + I(X_s)) = (X_1 \cup \ldots \cup X_{s-1}) \cap X_s$, and so by Lemma 5.2 it follows that $P_{I(X_1 \cup \ldots \cup X_{s-1}) + I(X_s)}$ has the same degree as $P_{(X_1 \cup \ldots \cup X_{s-1}) \cap X_s}$, i.e., the dimension of $(X_1 \cup \ldots \cup X_{s-1}) \cap X_s$ (which from (iii) is strictly smaller than the dimension of X_s). Thus in the above equality of polynomials, all of them have degree at most the dimension of X, which implies that either X_s or $X_1 \cup \ldots \cup X_{s-1}$ has the same dimension as X. We conclude the proof of (iv) from our induction hypothesis.

For (v), let r be the dimension of X. We know from (iii) that any chain as in the statement has length at most r. Hence it is enough to find a chain whose length is exactly r. We do it by induction on r. The case $r = 0$ is trivial, since we just need to take as X_0 any point of X. So assume $r \geq 1$. By (iv) we can find an irreducible component X_r of X of dimension r. Taking a homogeneous polynomial F not vanishing on X_r, we know from (ii) that $X_r \cap V(F)$ has dimension $r-1$, so by induction hypothesis there is a chain $X_0 \subsetneq \ldots \subsetneq X_{r-1}$ of irreducible sets contained in $X_r \cap V(F)$. Since clearly $X_{r-1} \subsetneq X_r$ (for they have different dimensions), we conclude the proof.

Finally, part (vi) is just Lemma 3.16, while part (vii) is a consequence of Proposition 3.15(iii), and part (viii) is an easy consequence of parts (ii) and (iv). $\square$

Observe that Proposition 5.7(v) shows that the dimension can be defined in purely topological terms. We can therefore extend the notion of dimension to quasiprojective sets.

Definition. The *dimension of a quasiprojective set* X is the maximum length r of a chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_r$ of irreducible closed subsets of X.

Lemma 5.8. *The dimension of a quasiprojective set is the dimension of its Zariski closure.*

Proof: We will just use Lemma 2.1 to see that a quasiprojective set and its closure have essentially the same chains of irreducible sets. So let $X \subset \mathbb{P}^n$ be a quasiprojective set and let $\overline{X}$ be its Zariski closure. Then $X = \overline{X} \cap U$ for some open set in $\mathbb{P}^n$. If $Z_0 \subsetneq Z_1 \subsetneq \ldots, \subsetneq Z_r$ is a chain of irreducible closed sets in X, then clearly $\overline{Z}_0 \subsetneq \overline{Z}_1 \subsetneq \ldots \subsetneq \overline{Z}_r$ is a chain of irreducible closed sets in $\overline{X}$. Hence $\dim X \leq \dim \overline{X}$.

For the other inequality we just need to be careful to take chains in $\overline{X}$ that do not "disappear" when intersecting with U. To this purpose, we will need to slightly adapt the proof of Proposition 5.7(v). So take an irreducible component Y of $\overline{X}$ of maximum dimension r and write $Z = Y \setminus U$. This is a closed subset of Y, and it is strictly contained in Y, since otherwise removing Y from the components of $\overline{X}$ we will obtained a closed subset containing X strictly smaller than $\overline{X}$. In particular, $\dim Z < \dim Y = r$.

With this set-up, we take a homogeneous polynomial F not vanishing neither on Y nor on any component of Z (for instance, if $Z \neq \emptyset$, we can choose a point at each component and take a hyperplane not passing through any of these points). We then have that $\dim Z \cap V(F) = \dim Z - 1$ and $\dim Y \cap V(F) = r - 1$. We take Y' to be an irreducible component of $Y \cap V(F)$ of dimension $r-1$, and $Z' = Y' \cap Z$, which has dimension at most $r-2$. Repeating the same process for Y' and Z' and iterating (the process will become trivial as soon as the "bad" subset Z becomes empty), we obtain a chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_r$ of irreducible sets of $\overline{X}$ all of them meeting U. Intersecting then with U we get that $\dim X \geq r$, which completes the proof. $\square$

Proposition 5.9. *A projective set $X \subset \mathbb{P}^n$ can be expressed as $V(F)$ (with F homogeneous polynomial of positive degree) if and only if any irreducible component of X has dimension $n-1$.*

Proof: Assume first that $X = V(F)$ and write $F = F_1^{a_1} \ldots F_r^{a_r}$, a decomposition of F into irreducible factors. Then $X = V(F_1) \cup \ldots \cup V(F_r)$ is the decomposition of X into irreducible components. We then just need that each of the $V(F_i)$'s has dimension $n-1$, but this is immediate from their Hilbert polynomial.

Assume now that all the irreducible components of X has dimension $n-1$. If we prove that each irreducible component has the form $V(F)$, then also X will be defined by one polynomial (just the product of the polynomials defining the components). Hence it is enough to prove that any irreducible variety of dimension $n-1$ is defined by one polynomial. So let $X \subset \mathbb{P}^n$ be a projective variety of dimension $n-1$. In particular, $I(X)$ is not zero, so that there exist a homogeneous polynomial of positive degree in $I(X)$. Since $I(X)$ is prime, one irreducible factor F of this polynomial should also belong to $I(X)$. We then have $X \subset V(F)$, and since $V(F)$ is irreducible and both X and $V(F)$ have dimension $n-1$, Proposition 5.7(iii) concludes the proof. $\square$

Definition. A *hypersurface* in $\mathbb{P}^n$ will be a projective set of the type $V(F)$, with F a homogeneous polynomial of positive degree. Equivalently, by Corollary 3.18, X is a hypersurface if $I(X)$ is generated by one polynomial.

We introduce now the second invariant, which the reader should also find intuitive. Let $X \subset \mathbb{P}^n$ a projective set of dimension r. If H_1 is the equation of a general hyperplane, we have observed that $X \cap V(H_1)$ has dimension $r-1$. Moreover, if $P_X(l) = al^r+$ lower degree terms, then the Hilbert polynomial of $I(X) + (H_1)$ is $P_X(l) - P_X(l-1) = ral^{r-1}+$ lower degree terms. Iterating the process, we then obtain that, for general linear forms $H_1, \ldots, H_r$, the intersection $X \cap V(H_1, \ldots, H_r)$ consists of a finite set of points, and that $I(X) + V(H_1, \ldots, H_r)$ has (constant) Hilbert polynomial $r!a$. As we have also remarked, this constant should count the number of points, maybe with some multiplicity (we will immediately make more precise this idea). It is then natural to make the following definitions.

Definition. The *degree of a projective set* $X \subset \mathbb{P}^n$ of dimension r is $\deg X = ar!$, where a is the coefficient of l^r in the Hilbert polynomial of P_X.

Exercise 5.10. Prove that the degree of a hypersurface $X \subset \mathbb{P}^n$ is the degree of any polynomial generating $I(X)$.

Exercise 5.11. Prove that any linear subspace has degree 1.

Exercise 5.12. Prove that the degree of a Veronese variety (image of $\mathbb{P}^n$ under ν_d) is d^n, and the degree of a Segre variety is $\binom{n+m}{n} = \binom{n+m}{m}$. In particular, what we called rational normal curve of degree d has indeed degree d.

Lemma 5.13. *The degree satisfies the following properties:*

(i) The degree of a union of two projective sets of different dimensions is the degree of the set of maximum dimension.

(ii) The degree of the union of two projective sets of the same dimension is the sum of the degrees of the two sets, provided that the two sets do not share components of maximal dimension.

Proof: The proof is based on the exact sequence in Lemma 3.1

$$0 \to S(X \cup Y) \to S(X) \oplus S(Y) \to S/(I(X) + I(Y))$$

If for instance $\dim X > \dim Y$ then the Hilbert polynomials of Y and $I(X) + I(Y)$ have degree strictly smaller than the degree of P_X (which is the dimension of X). Hence the leading coefficients of the Hilbert polynomials of $X \cup Y$ and X (which have the same dimension) are the same. As a consequence, $X \cup Y$ and X have the same degree, proving (i).

If X and Y have now the same dimension, the hypothesis in (ii) implies that $X \cap Y$ has dimension strictly less than $\dim X = \dim Y$. Therefore, the Hilbert polynomial of $I(X) + I(Y)$ has degree strictly less than $\dim X = \dim Y$. Using the same exact sequence we see that the leading coefficient of $P_{X \cup Y}$ is the sum of the leading coefficients of P_X and P_Y, which completes the proof of the lemma. $\square$

Definition. The *multiplicity of intersection of two projective sets* $X, Y \subset \mathbb{P}^n$ at a point p (assuming that p is an irreducible component of $X \cap Y$) is the value (which is constant by Remark 5.3) of the Hilbert polynomial of the $I(p)$-primary component of $I(X) + I(Y)$ (note that from Theorem 2.7 the $I(p)$-primary component is independent on the primary decomposition, since the fact that p is an irreducible component means that $I(p)$ is a minimal prime).

The following proposition shows (according to the philosophy of Example 1.21) that two projective sets meet with multiplicity bigger than one at some point iff they share some infinitesimal direction at the point

Proposition 5.14. *Let $p \in \mathbb{P}^n$ be a point and I an $I(p)$-primary ideal different from $I(p)$. Then it is possible to choose homogeneous coordinates in $\mathbb{P}^n$ such that $p = (1 : 0 : \ldots : 0)$ and $I \subset (X_1^2, X_2, \ldots, X_n)$. In particular, an $I(p)$-primary ideal I has Hilbert polynomial $P_I = 1$ if and only if $I = I(p)$.*

Proof: We can clearly assume $p = (1 : 0 : \ldots : 0)$ in some suitable reference, i.e. $\sqrt{I} = I(p) = (X_1, \ldots, X_n)$. We thus have $(X_1, \ldots, X_n)^d \subset I$ for some $d \in \mathbb{N}$. Since by hypothesis $I \subsetneq \sqrt{I}$, the minimum integer d satisfying the above inclusion is at least 2. We can thus find a monomial of degree $d - 1$ in the variables $X_1, \ldots, X_n$ that is not in I. This immediately implies that we can find a homogeneous polynomial $H \in \mathbb{K}[X_1, \ldots, X_n]$ of degree $d - 2$ such that its product by at least one of the variables $X_1, \ldots, X_n$ is not in I.

We consider now the vector space V of the linear forms in the variables $X_1, \ldots, X_n$. Inside this vector space, let W be the (proper) linear subspace consisting of the forms whose multiplication with H is in I. After changing coordinates in V if necessary, we can assume that $X_r, \ldots, X_n$ form a basis of W, for some r that is necessarily at least 2. Let us see that with this choice of coordinates we have the wanted inclusion $I \subset (X_1^2, X_2, \ldots, X_n)$.

Let F be any homogeneous polynomial in the ideal I and let m be its degree. We can write $F = A_0 X_0^m + A_1 X_0^{m-1} + \ldots + A_m$, where each A_i is a homogeneous polynomial of degree i in the variables $X_1, \ldots, X_n$. Obviously the constant A_0 is zero, since F must vanish at p, which is $(1 : 0 : \ldots : 0)$ in our coordinates. The wanted inclusion will hence follow at once if we show that the linear form A_1 does not depend on X_1. For this, it is enough to prove that A_1 is in W, i.e. that $A_1 H$ is in I. Since I is $(X_1, \ldots, X_n)$-primary and X_0^{m-1} is not in $(X_1, \ldots, X_n)$, this is equivalent to prove that $X_0^{m-1} A_1 H$ is in I. This follows now immediately from the equality $X_0^{m-1} A_1 H = HF - X_0^{m-2} A_2 H - \ldots - A_m H$, after observing that each $A_i H$ on the right-hand side has degree at least d, so it is in $(X_1, \ldots, X_n)^d$, hence in I. This completes the proof. □

The main geometrical meaning of the degree (which will be improved in Theorem 12.1(ii)) is the following.

Theorem 5.15. *Let $X \subset \mathbb{P}^n$ be a projective set of dimension r and let $H_1, \ldots, H_r$ be linear forms such that for each $i = 1, \ldots, r$ the hyperplane $V(H_i)$ does not contain any irreducible component of $X \cap V(H_1, \ldots, H_{i-1})$. Then $\deg X$ is the sum of the intersection multiplicities of X and $V(H_1, \ldots, H_r)$ at their points of intersection.*

Proof: From our hypothesis and Proposition 3.15(v), we get that $I(X) + (H_1, \ldots, H_r)$ has Hilbert polynomial equal to $\deg X$. But on the other hand, if $I_1, \ldots, I_s$ are the non-embedded components of $I(X) + (H_1, \ldots, H_r)$, then the points of intersection of X and $V(H_1, \ldots, H_r)$ are $p_1 = V(I_1), \ldots, p_s = V(I_s)$ and the intersection multiplicity of X and

$V(H_1, \ldots, H_r)$ at p_i is precisely P_{I_i}. Observe that if there is an embedded component I' of $I(X) + (H_1, \ldots, H_r)$, then its radical must strictly contain some $\sqrt{I_i}$. Therefore $V(I') \subsetneq V(I_i) = \{p_i\}$, which implies that $V(I') = \emptyset$ and I' is thus necessarily $\mathfrak{M}$-primary. Hence from Proposition 3.15(ii) we obtain that $P_{I(X)+(H_1,\ldots,H_r)} = P_{I_1 \cap \ldots \cap I_s}$. Having in mind that each $I_i + I_j$ defines $\{p_i\} \cap \{p_j\} = \emptyset$ (if $i \neq j$), and thus its Hilbert polynomial is zero, we can use Proposition 3.15(iv) to conclude that $P_{I_1 \cap \ldots \cap I_s} = P_{I_1} + \ldots + P_{I_s}$, which completes the proof. □

Exercise 5.16. Prove Bézout's theorem for plane curves: If C_1 and C_2 are plane curves not having common components and of respective degrees d_1 and d_2, then $C_1 \cap C_2$ meet in a finite number of points, which are exactly $d_1 d_2$ counted with multiplicity.

Exercise 5.17. More generally, prove that if $X \subset \mathbb{P}^n$ is a projective set of dimension r and degree d, and if $X_1, \ldots, X_r$ are hypersurfaces of respective degrees $d_1, \ldots, d_r$ such that $X \cap X_1 \cap \ldots, \cap X_r$ is a finite number of points, then the number of points, counted with multiplicity, is $dd_1 \ldots d_r$. Observe that, in particular, this implies that, if $I(X) + (F)$ is nice enough (for instance if it is a radical ideal and $X \cap V(F)$ has dimension $r - 1$) the intersection of X with the hypersurface $V(F)$ has degree $d \deg F$. We will strengthen this result in Proposition 10.10.

Observe that we have seen (Exercise 5.12) that the rational normal curve $C = \{(t_0^n : \ldots : t_n^n)\} \subset \mathbb{P}^n$ has degree n. On the other hand, if we intersect C with a hypersurface $X \subset \mathbb{P}^n$ of degree d, then from Exercise 5.17 it follows that $C \cap X$ consists of nd points counted with multiplicity. This is intuitively clear, since $I(X)$ is generated by a homogeneous polynomial $F \in \mathbb{K}[X_0, \ldots, X_n]$ of degree d, and the intersection of C and X comes from the solutions in $\mathbb{P}^1$ of the polynomial $F(t_0^n, \ldots, t_n^n)$. But this is a homogeneous polynomial of degree nd, so it has nd roots counted with multiplicity.

In a similar way, the set $\{(t_0^4 : t_0^3 t_1 : t_0 t_1^3 : t_1^4)\} \subset \mathbb{P}^3$ should be a curve of degree four (as can be deduced as once from Exercise 3.11), since the intersection with a general plane comes from the roots of a homogenous polynomial of degree four. However in both cases we do not know a priori that the two notions of multiplicity (as defined here and the multiplicity of a root) coincide. They indeed coincide, but the proof its too technical to be done here.

We finally introduce briefly the last main invariant of a projective set. It could seem quite artificial at a first glance. However, the reader who is familiar with the theory of algebraic curves (maybe under the name of Riemann surfaces) or just with the theory of plane curves will find it more natural after Exercise 5.18. Moreover, we will see later on that the arithmetic genus is in fact an intrinsic invariant of projective sets (Theorem 8.15).

Definition. Let $X \subset \mathbb{P}^n$ be a projective set. The *arithmetic genus* of X is the number $p_a(X) := (-1)^r(P_X(0) - 1)$, where r is the dimension of X.

Exercise 5.18. Prove that the arithmetic genus of a plane curve of degree d is $(d-1)(d-2)/2$.

Exercise 5.19. Prove that any Segre or Veronese variety has arithmetic genus zero.

The exact sequence in Lemma 3.1 allows to express the arithmetic genus of a union of two projective sets $X, Y \subset \mathbb{P}^n$ in terms of the arithmetic genus of X, Y and $X \cap Y$, provided that this intersection is nice. The best situation is of course when $I(X \cap Y) = I(X) + I(Y)$. The following exercises are a sample of how to deal with this method.

Exercise 5.20. Let $C, D \subset \mathbb{P}^n$ be two curves meeting at a points counted with multiplicity. Prove that the arithmetic genus of $C \cup D$ is $p_a(C) + p_a(D) + a - 1$.

Exercise 5.21. Defining the arithmetic genus of any homogeneous ideal by just considering its Hilbert polynomial, prove that the ideal $I_0 = (X_3^2, X_1X_3, X_2X_3, X_1X_2)$ (which appeared in Example 1.23) has arithmetic genus -1. Conclude that the arithmetic genus of all the ideals I_t (in the same exercise) is preserved, while the arithmetic genus of $V(I_t) = L \cup L_t)$ varies (depending on whether $t = 0$ or not). Hence the arithmetic genus is preserved when one considers ideals instead of the varieties they define. This shows why sometimes embedded components are needed.

6. Product of varieties

We will need to extend all the notions we have seen to products of projective spaces. We first observe that via the Segre map we can still regard the product of two projective spaces as a projective set. This can be clearly extended inductively to the product of an arbitrary number of projective spaces, but we will restrict our attention to just the product of two, not only because an easy iteration takes care of the general case, but also and essentially that this will be the case that will be of interest to us.

It is usually better to use intrinsic equations inside products of two projective spaces, and clearly these equations must be bihomogeneous (in the case of two projective spaces). The following result states that this way of defining subsets is equivalent to defining projective sets.

Proposition 6.1. *Consider $\mathbb{P}^n \times \mathbb{P}^m$ as a subset of $\mathbb{P}^{nm+n+m}$ via the Segre map. Then any projective subset of $\mathbb{P}^{nm+n+m}$ that is contained in $\mathbb{P}^n \times \mathbb{P}^m$ is defined as the zero locus of a set of bihomogeneous polynomials. Reciprocally, any subset of $\mathbb{P}^n \times \mathbb{P}^n$ defined as the zero locus of a set of bihomogeneous polynomials is a projective set in $\mathbb{P}^{nm+n+m}$.*

Proof: Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be a projective set of $\mathbb{P}^{nm+n+m}$ defined by homogeneous polynomials $F_1, \ldots, F_s \in \mathbb{K}[Z_{00}, Z_{01}, \ldots, Z_{nm}]$. For each F_l ($l = 1, \ldots, s$), the substitutions $Z_{ij} = X_i Y_j$ provides a polynomial $G_l \in \mathbb{K}[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$, which is bihomogeneous of bidegree (d_l, d_l) if $d_l = \deg F_l$. It is then clear that X is defined in $\mathbb{P}^n \times \mathbb{P}^m$ as the zero locus of these polynomials.

And reciprocally, suppose $X \subset \mathbb{P}^{nm+n+m}$ is defined by bihomogeneous polynomials. First of all, observe that a bihomogeneous polynomial $G \in \mathbb{K}[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ of bidegree (a, b), with say $a < b$, vanishes exactly at the same locus as the polynomials $X_0^{b-a}G, \ldots, X_n^{b-a}G$. Hence we can assume that X is defined by bihomogeneous polynomials $G_1, \ldots, G_s \in \mathbb{K}[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ such that each G_l has bidegree (d_l, d_l). It is then clear that each G_l can be obtained from a (not necessarily unique) homogeneous polynomial $F_l \in \mathbb{K}[Z_{00}, Z_{01}, \ldots, Z_{nm}]$ of degree d_l by means of the substitutions $Z_{ij} = X_i Y_j$. Then X, as a subset of $\mathbb{P}^{nm+n+m}$ is defined by the polynomials $F_1, \ldots, F_s$ plus the equations of the Segre variety (observe that it is strictly necessary to add these equations). This completes the proof. $\square$

Exercise 6.2. Identifying $V(X_0X_3 - X_1X_2)$ with the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, find bihomogeneous equations of the twisted cubic (Example 1.10) and the curve of Exercise 1.14.

We include now as exercises a few representative examples of subsets inside products that are in fact projective sets. Later on we will not prove (maybe neither mention) that similar subsets are closed, hoping that the reader will have acquired the feeling that naturally defined (in an algebraic way) subsets are always closed.

Exercise 6.3. Prove that the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$ is a closed set.

Exercise 6.4. Prove that, for any choice of integers k, l, s, the subset of $\mathbb{G}(k,n) \times \mathbb{G}(l,n)$ consisting of the pairs (Λ, Ω) such that $\dim(\Lambda \cap \Omega) \geq s$ is a closed set (when considering each Grassmannian inside its corresponding projective space via the Plücker embedding).

Exercise 6.5. Identifying $\mathbb{P}^{\binom{n+d}{d}-1}$ with the set of hypersurfaces of degree d in $\mathbb{P}^n$, prove that the subset of $\mathbb{P}^{\binom{n+d}{d}-1} \times \mathbb{G}(k,n)$ consisting of pairs (X, Λ) for which $\Lambda \subset X$ is a projective set.

We can now translate to the bihomogeneous case all the definitions and results that we had for the homogeneous case.

Definition. The *bihomogeneous ideal of a subset* $X \subset \mathbb{P}^n \times \mathbb{P}^m$ will be the ideal $\overline{I}(X)$ generated by all the bihomogeneous polynomials vanishing at all points of X.

Notation. Since we are going to use them very often, we will usually write, when no confusion arises, $S = \mathbb{K}[Z_{00}, Z_{01}, \ldots, Z_{nm}]$, $\overline{S} = \mathbb{K}[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ and will denote by Φ to the map $S \to \overline{S}$ defined by the substitutions $Z_{ij} = X_i Y_j$. We will use a bar to indicate that an ideal is in $\overline{S}$. We will write $\overline{\mathfrak{M}}_1 = (X_0, \ldots, X_n)$ and $\overline{\mathfrak{M}}_2 = (Y_0, \ldots, Y_m)$.

We remark that, with the above notation, if $X \subset \mathbb{P}^n \times \mathbb{P}^m$, then its homogeneous ideal $I(X)$ is $\Phi^{-1}(\overline{I}(X))$. It is not true however that $\overline{I}(X)$ is the ideal generated by $\Phi(I(X))$ (as Exercise 6.2 shows). All this is implicit in the proof of Proposition 6.1.

Proposition 6.6. *Bihomogeneous ideals satisfy the following properties:*

(i) Every bihomogeneous ideal is a finite intersection of bihomogeneous primary ideals. The minimal primes in this decomposition that do not define the empty set provides the decomposition of $V(\overline{I})$ into irreducible components.

(ii) A bihomogeneous ideal $\overline{I}$ defines the empty set if and only if $\overline{I}$ contains some power of $\overline{\mathfrak{M}}_1 \overline{\mathfrak{M}}_2$.

(iii) For any bihomogeneous ideal $\overline{I}$ there exists a polynomial $\overline{P}_{\overline{I}} \in \mathbb{Q}[T_1, T_2]$ such that for $l_1, l_2 >> 0$ it holds that $\overline{P}_{\overline{I}}(l_1, l_2)$ is the dimension as a $\mathbb{K}$-vector space of the quotient $(\overline{S}/\overline{I})_{l_1, l_2}$.

Proof: The proof of the primary decomposition goes exactly as in the homogeneous case (Lemmas 2.4 and 2.5), by just observing that all the ideals involved now in the proof are bihomogeneous. One also proves as in Lemma 2.2 that $V(\overline{I})$ is irreducible if and only if $\overline{I}$

is prime, so that the second part of (i) comes from the uniqueness of the decomposition of a projective set into irreducible components.

As for (ii), it is clear that any ideal containing a power of $\overline{\mathfrak{M}}_1\overline{\mathfrak{M}}_2$ defines the empty set. For the converse, assume that no power of $\overline{\mathfrak{M}}_1\overline{\mathfrak{M}}_2$ is contained in $\overline{I}$. Then we can assume after changing coordinates that no power of X_0Y_0 is contained in $\overline{I}$. Consider now $I = \Phi^{-1}(\overline{I})$. Then no power of Z_{00} is contained in I. But then the weak Nullstellensatz (Theorem 1.24) implies that $V(I) \neq \emptyset$. Since $V(I) = V(\overline{I})$, this completes the proof of (ii).

To prove (iii), we first observe that, as in the homogeneous case, parts (i) and (ii) allow us to remove from the primary decomposition of $\overline{I}$ all the ideals defining the empty set. We will prove the existence of the polynomial by induction on $\dim V(\overline{I})$. As just remarked, if $V(\overline{I}) = \emptyset$ there is nothing to prove since we can take $\overline{P}_{\overline{I}} = 0$. So assume $V(\overline{I}) \neq \emptyset$. In particular, we can find a linear form $H \in \mathbb{K}[X_0, \dots, X_n]$ such that no irreducible component of $V(J)$ is contained in $V(H)$. Therefore the dimension of $V(\overline{I}) \cap V(H)$ is strictly smaller than the dimension of $V(\overline{I})$ (by Proposition 5.7(iii)). But we have also an exact sequence

$$0 \to \overline{S}/\overline{I}(-1,0) \xrightarrow{\cdot X_0} \overline{S}/\overline{I} \to \overline{S}/(\overline{I} + (H)) \to 0$$

(obtained as in Lemma 3.12). We can use now the induction hypothesis to obtain a polynomial $Q \in \mathbb{Q}[T_1, T_2]$ measuring the dimension of $(\overline{S}/(\overline{I} + (H)))_{l_1,l_2}$ for large values of l_1, l_2. On the other hand, if $I = \Phi^{-1}(\overline{I})$ we have that $(S/I)_l$ is isomorphic, via Φ, to $(\overline{S}/\overline{I})_{l,l}$. Therefore (iii) will hold if and only if there exists a polynomial $\overline{P}_{\overline{I}} \in \mathbb{Q}[T_1, T_2]$ satisfying the conditions:

$$\overline{P}_{\overline{I}}(T_1, T_2) = \overline{P}_{\overline{I}}(T_1 - 1, T_2) + Q(T_1, T_2)$$

$$\overline{P}_{\overline{I}}(T, T) = P_I(T)$$

(where P_I is the Hilbert polynomial of I). Writing Q in the form $Q(T_1, T_2) = A_0(T_2)\binom{T_1}{0} + A_1(T_2)\binom{T_1}{1} + \ldots + A_d(T_2)\binom{T_1}{d}$ (recall the proof of Theorem 3.13) we see that $\overline{P}_{\overline{I}}$ must take the form (in order to satisfy the first property) $\overline{P}_{\overline{I}}(T_1, T_2) = C(T_2) + A_0(T_2)\binom{T_1}{1} + A_1(T_2)\binom{T_1}{2} + \ldots + A_d(T_2)\binom{T_1}{d+1}$ for some $C \in \mathbb{Q}[T]$. But now the second condition is equivalent to $P_I(T) = C(T) + A_0(T)\binom{T}{1} + A_1(T)\binom{T}{2} + \ldots + A_d(T)\binom{T}{d+1}$, which univoquely determines C and hence $\overline{P}_{\overline{I}}$. This completes the proof of the proposition. □

Definition. The polynomial $\overline{P}_{\overline{I}}$ whose existence has just been proved will be called the *Hilbert polynomial of the bihomogeneous ideal* $\overline{I}$.

Exercise 6.7. Find the Hilbert polynomial of the bihomogeneous ideals of the curves in Exercise 6.2.

Exercise 6.8. Translate to the bihomogeneous case the primary decomposition of Exercise 2.10.

Remark 6.9. In contrast with the homogeneous case, in which there was at most only one primary component defining the empty set (with radical $\mathfrak{M}$), this does not happen in the bihomogeneous case. From Proposition 6.6(ii), if a primary ideal defines the empty set then its radical contains $\overline{\mathfrak{M}}_1 \cap \overline{\mathfrak{M}}_2$. Since the radical is prime then we only get that the radical of this primary ideal contains either $\overline{\mathfrak{M}}_1$ or $\overline{\mathfrak{M}}_2$. By abuse of notation, will say that a primary bihomogeneous ideal is $\overline{\mathfrak{M}}_i$*-primary* ($i = 1, 2$) if its radical contains $\overline{\mathfrak{M}}_i$. With this definition, a bihomogeneous primary ideal defines the empty set if and only if it is $\overline{\mathfrak{M}}_1$-primary or $\overline{\mathfrak{M}}_2$-primary.

Example 6.10. To see an example of how the above remark affects to the primary decomposition in the bihomogeneous case, consider in $\mathbb{K}[t_0, t_1, s_0, s_1]$ the ideal $I = (t_0 s_1^3 - t_1 s_0^3, t_0 s_1 - t_1 s_0)$. It can be proved that

$$I = (t_0, s_0) \cap (t_1, s_1) \cap (t_0 - t_1, s_0 - s_1) \cap (t_0 + t_1, s_0 + s_1) \cap (t_0, t_1) \cap (t_0 s_1 - t_1 s_0, s_0^3, s_0^2 s_1, s_0 s_1^2, s_1^3)$$

Obviously each of the first four ideals represents a point while the last two ideals, even if they are minimal, represent the empty set (that is the reason of the apparently redundant condition in the statement of Proposition 6.6(i)). In fact, it can be seen that the above is the unique primary decomposition for I (for instance consider I as a homogeneous ideal rather than bihomogeneous, and then it has no embedded components, so it is possible to use Theorem 2.7). To see the geometry of this example, the reader should have solved correctly Exercise 6.2 and then take a look at Example 1.22 and Exercise 2.13.

In the same way as in the projective case, we have the following properties of the Hilbert polynomial (some of them have been already proved implicitly).

Proposition 6.11. *Let $\overline{I} \subset \mathbb{K}[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ be a bihomogeneous ideal and let $X = V(\overline{I}) \subset \mathbb{P}^n \times \mathbb{P}^m$ be the corresponding projective set.*

- *(i) If $F \in \mathbb{K}[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ is a bihomogeneous polynomial of bidegree (a, b) not vanishing on any irreducible component of X, then the Hilbert polynomial of $\overline{J} = \overline{I} + (F)$ is given by $\overline{P}_{\overline{J}}(T_1, T_2) = \overline{P}_{\overline{I}}(T_1, T_2) - \overline{P}_{\overline{I}}(T_1 - a, T_2 - b)$.*
- *(ii) If $I = \Phi^{-1}(\overline{I})$, then $P_I(T) = \overline{P}_{\overline{I}}(T, T)$. In particular, $P_X(T) = \overline{P}_X(T, T)$.*
- *(iii) If the total degree of $\overline{P}_{\overline{I}}$ is r, then the coefficient of $T_1^i T_2^{r-i}$ is non-negative for any $i = 0, \ldots, d$.*
- *(iv) The (total) degree of $\overline{P}_{\overline{I}}$ is the dimension of X.*
- *(v)* (Bihomogeneous Nullstellensatz) *If I is a bihomogeneous ideal not defining the empty set, then $\overline{I}V(I) = \sqrt{I}$.*

Proof: Part (i) is an immediate consequence of the exact sequence

$$0 \to \overline{S}/\overline{I}(-a,-b) \xrightarrow{\cdot F} \overline{S}/\overline{I} \to \overline{S}/\overline{J} \to 0$$

Part (ii) was already proved when showing the existence of the Hilbert polynomial in Proposition 6.6(iii). Part (iii) is deduced iterating (i), since adding to $\overline{I}$ suitable bihomogeneous forms, i of them of bidegree $(1,0)$ and $r-i$ of bidegree $(0,1)$ we get an ideal whose Hilbert polynomial is constant and equal to $i!(r-i)!$ times the coefficient. Part (iv) is an easy consequence of (ii) and (iii). And finally the proof of part (v) goes exactly as the one for Theorem 3.17 and we leave it as an exercise. □

Definition. Given a projective set $X \subset \mathbb{P}^n \times \mathbb{P}^m$ of dimension r, we will call the *multidegree* of X to the $(r+1)$-uple of integers $(d_0, \dots, d_r)$ such that the homogeneous part of degree r of $\overline{P}_X$ is $\sum_{i=0}^{r} \frac{d_i}{i!(r-i)!} T_1^i T_2^{r-i}$. The number d_i (which is non-negative by part (iii) of the above Proposition) represents the number of points, "counted with multiplicity", in the intersection of X with the pull-backs of sufficiently general linear subspaces, one of $\mathbb{P}^n$ of codimension i and another one of $\mathbb{P}^m$ of codimension $r-i$.

Exercise 6.12. Prove that a projective set of $\mathbb{P}^n \times \mathbb{P}^m$ has the form $V(F)$ if and only if all its irreducible components have dimension $n+m-1$. Moreover, if F is square-free of bidegree (a,b), then $V(F)$ has bidegree (a,b).

Exercise 6.13. Prove that the arithmetic genus of a curve of bidegree (a,b) in $\mathbb{P}^1 \times \mathbb{P}^1$ is $(a-1)(b-1)$.

Proposition 6.14. *Let $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ be two projective sets. Then the Hilbert polynomial of $X \times Y$ in $\mathbb{P}^n \times \mathbb{P}^m$ is given by $\overline{P}(T_1,T_2) = P_X(T_1)P_Y(T_2)$. As a corollary, $\dim(X \times Y) = \dim X + \dim Y$.*

Proof: Take $l_1, l_2 \in \mathbb{N}$ large enough so that $\dim S(X)_{l_1} = P_X(l_1)$ and $\dim S(Y)_{l_2} = P_Y(l_2)$. Call these dimensions respectively s and t. Take $A_1, \dots, A_s \in \mathbb{K}[X_0, \dots, X_n]_{l_1}$ forming a basis modulo $I(X)$, and $B_1, \dots, B_t \in \mathbb{K}[Y_0, \dots, Y_m]_{l_2}$ forming a basis modulo $I(Y)$. We first remark that any polynomial in $I(X)$ or $I(Y)$, regarded as a polynomial in $\mathbb{K}[X_0, \dots, X_n, Y_0, \dots, Y_m]$, clearly belongs to $\overline{I}(X \times Y)$.

Now, any homogeneous polynomial in $\mathbb{K}[X_0, \dots, X_n, Y_0, \dots, Y_m]$ of bidegree (l_1, l_2) can be written in the form $\sum P_\alpha Q_\alpha$ with each $P_\alpha \in \mathbb{K}[X_0, \dots, X_n]$ homogeneous of degree l_1 and $Q_\alpha \in \mathbb{K}[Y_0, \dots, Y_m]$ homogeneous of degree l_2 (e.g. decomposing the polynomial into monomials). But now each P_α is a linear combination, modulo $I(X)$, of $A_1, \dots, A_s$, while each Q_α is a linear combination, modulo $I(Y)$, of $B_1, \dots, B_t$. Therefore, the classes of the products A_iB_j ($i = 1, \dots, s$, $j = 1, \dots, t$) generate $\overline{S}(X \times Y)$ as a vector space over

$\mathbb{K}$. The proposition will be proved as soon as we prove that these generators are linearly independent.

Hence assume that there exist $\lambda_{ij} \in \mathbb{K}$ ($i = 1, \ldots, s$ and $j = 1, \ldots, t$) such that $\sum_{i,j} \lambda_{ij} A_i B_j \in \overline{I}(X \times Y)$. Fix for a while $(x_0 : \ldots : x_n) \in X$. Then we have that the polynomial $\sum_{i,j} \lambda_{ij} A_i(x_0, \ldots, x_n) B_j \in \mathbb{K}[Y_0, \ldots, Y_m]$ (which is homogeneous of degree l_2 belongs to $I(Y)$. But this yields a linear relation of the classes of $B_1, \ldots, B_t$ in $S(Y)$ with coefficients $\sum_i \lambda_{i1} A_i(x_0, \ldots, x_n), \ldots, \sum_i \lambda_{it} A_i(x_0, \ldots, x_n)$. Since these classes were linearly independent (in fact a basis), we deduce that $\sum_i \lambda_{ij} A_i(x_0, \ldots, x_n) = 0$ for all $j = i \ldots, t$. Since this was done for an arbitrary point $(x_0 : \ldots : x_n) \in X$, it follows that the polynomials $\sum_i \lambda_{ij} A_i \in \mathbb{K}[X_0, \ldots, X_n]$ (homogeneous of degree l_1) belong to $I(X)$ for each $j = 1, \ldots, t$. But again this provides linear relations of the classes of $A_1, \ldots, A_s$ in $S(X)$, from which we obtain $\lambda_{ij} = 0$ for $i = 1, \ldots, s$ and $j = 1, \ldots, t$. This proves that the classes of the products $A_i B_j$ are independent and concludes the proof of the lemma. $\square$

Exercise 6.15. In the situation of the above proposition, prove that $I(X \times Y)$ is generated by the polynomials in $I(X)$ and $I(Y)$.

Remark 6.16. For those who are familiar with the tensor product, the proof of the above proposition is just showing that $\overline{S}(X \times Y)_{l_1, l_2}$ is isomorphic to $S(X)_{l_1} \otimes S(Y)_{l_2}$ (via the natural isomorphism $\mathbb{K}[X_0, \ldots, X_n] \otimes \mathbb{K}[Y_0, \ldots, Y_m] \cong \mathbb{K}[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$).

7. Regular maps

Before making the rigorous definition of which maps we should allow between projective sets, let us study some examples to see what it is natural to require. The only maps we have seen so far are the Segre and Veronese embeddings. To be consequents with their names we should even allow them to be isomorphisms onto their images. Let us see what consequences this would have.

Example 7.1. The Veronese maps are well-defined because they are defined by homogeneous polynomials of the same degree. This could look a good definition for a morphism, but let us see that this is not the case. If we really want a Veronese embedding to be an isomorphism onto its image, we should be able to define a reasonable inverse. Consider the easiest nontrivial example, i.e. the double Veronese embedding of $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$ given by $(t_0 : t_1) \mapsto (t_0^2 : t_0t_1 : t_1^2)$. The inverse over the image $X = V(X_0X_2 - X_1^2)$ can be defined by $(X_0 : X_1 : X_2) \mapsto (X_0 : X_1)$ when $t_0 \neq 0$ (i.e. $X_0 \neq 0$ on X), while for $t_1 \neq 0$ (i.e. $X_2 \neq 0$) the definition could be $(X_0 : X_1 : X_2) \mapsto (X_1 : X_2)$. Both definitions coincide in the intersection of the two open sets $D(X_0) \cap D(X_2) \cap X$, just because the equation of X says precisely that $(X_0 : X_1) = (X_1 : X_2)$. And on the other hand, the two open sets cover the whole X. As a consequence, we do not have a global definition for the projection, but it is possible to cover X by open sets in such a way that the map has a definition on those open sets by homogeneous polynomials.

Example 7.2. About the Segre embedding, it does not have a priori much sense to talk about isomorphism, since precisely the algebraic structure of a product is given by this embedding. But however we can extract some conclusion. For instance, consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $\varphi_{1,1}((s_0 : s_1), (t_0 : t_1)) = (s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1))$. Its image in $\mathbb{P}^3$ is the quadric $Q := V(X_0X_3 - X_1X_2)$. As we said, we intend to use this Segre embedding to view Q as the product $\mathbb{P}^1 \times \mathbb{P}^1$. It is then a natural aspiration to want the projection maps to be considered as "allowable" maps. So let us see how they can be described. The first projection, for instance, can be defined by $(X_0 : X_1 : X_2 : X_3) \mapsto (X_0 : X_2)$, but this definition works only when $t_0 \neq 0$, i.e. outside the line $V(X_0, X_2)$ (which is contained in Q). But if $t_0 = 0$, then $t_1 \neq 0$, and we can defined the second projection by $(X_0 : X_1 : X_2 : X_3) \mapsto (X_1 : X_3)$. So we find again two alternative definitions of the map, one outside $V(X_0, X_2)$ and the other one outside $V(X_1, X_3)$. And again both definitions coincide in the intersection of the two open sets (which cover Q) because the equation of Q says that $(X_0 : X_2) = (X_1 : X_3)$.

Definition. A map $f : X \to Y$ between quasiprojective sets $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ is said to be a *regular map* (or simply a *morphism*) if every point in X has a neighborhood $U \subset X$ such that $f(p) = (F_0(p) : \ldots : F_m(p))$ for any $p \in U$ (where $F_0, \ldots, F_m \in \mathbb{K}[X_0, \ldots, X_n]$

are homogeneous polynomials of the same degree not vanishing simultaneously at any point of U). A bijective regular map f such that f^{-1} is also regular is called an *isomorphism*.

Example 7.3. It is not true in general that a bijective regular map is an isomorphism. Consider the cuspidal cubic $X = V(X_0X_2^2 - X_1^3) \subset \mathbb{P}^2$ and the map $f : \mathbb{P}^1 \to X$ defined by $f(t_0 : t_1) = (t_0^3 : t_0t_1^2 : t_1^3)$. It is clearly a regular bijective map. Let us see that however its inverse is not regular. First of all, in the open set $X_2 \neq 0$ the inverse can be defined by $f^{-1}(x_0 : x_1 : x_2) = (x_1 : x_2)$. Assume for contradiction that in a neighborhood of $(1 : 0 : 0)$ we can find $F_0, F_1 \in \mathbb{K}[X_0, X_1, X_2]$ homogeneous of the same degree, say d, such that $f^{-1}(x_0 : x_1 : x_2) = (F_0(x_0, x_1, x_2) : F_1(x_0, x_1, x_2))$ (and hence in particular $F_0(1, 0, 0) \neq 0$ since the inverse image of $(1 : 0 : 0)$ is $(1 : 0)$). But then it holds that $X_1F_1 - X_2F_0$ vanishes on an open set of X, and hence on the whole X. In other words, $X_1F_1 - X_2F_0$ belongs to $I(X)$. By Exercise 1.13, we have that $X_1F_1 - X_2F_0 = A(X_0X_2^2 - X_1^3)$ for some homogeneous polynomial A. We can rewrite this equality as

$$X_1(F_1 + X_1^2) = X_2(F_0 + X_0X_2)$$

from which we deduce that $F_1 = -X_1^2 + BX_2$ and $F_0 = -X_0X_2 + BX_1$, for some homogeneous polynomial B. In particular $F_0(1, 0, 0) = A_0(1, 0) = 0$, which is absurd. Hence f^{-1} is not a morphism and f is not an isomorphism. In fact we will see later (see Theorem 8.15) that $\mathbb{P}^1$ and X are not isomorphic).

As we did for closed sets inside products, we invite the reader to check by hand in the following exercises that the proposed maps are regular, and to believe without prove (but having the right intuition) that others maps that will appear later on are also regular. Of course the reader is free to refuse the invitation and check carefully the regularity of all the maps throughout the notes.

Exercise 7.4. Prove that the Segre and Veronese maps are isomorphisms.

Exercise 7.5. Prove that the projection of $\mathbb{P}^n \times \mathbb{P}^m$ over any of its two factors is a regular map.

Exercise 7.6. Prove that the linear projection from a linear subspace $\Lambda \subset \mathbb{P}^n$ of dimension r defines a regular map $\mathbb{P}^n \setminus \Lambda \to \mathbb{P}^{n-r-1}$. More generally, if Z is the set of k-spaces meeting Λ, prove also that the map $\mathbb{G}(k, n) \setminus Z \to \mathbb{G}(k, n - r - 1)$ is regular. The following exercise will show how to define in an intrinsic way a linear projection.

Exercise 7.7. Let Λ be a linear of $\mathbb{P}^n$ of dimension r and let Z be the set of k-spaces meeting Λ.

(i) Prove that the map $\mathbb{G}(k, n) \setminus Z \to \mathbb{G}(k + r + 1, n)$, obtained by taking the linear span with Λ, is a morphism.

(ii) If Λ' is a linear space of $\mathbb{P}^n$ of codimension $r+1$, and Z' is the set of k'-spaces containing Λ, then prove that the map $Z' \to \mathbb{G}(k'-r-1, \Lambda')$ (associating to each k'-space containing Λ its intersection with Λ') is an isomorphism.

Lemma 7.8. *Let $f : X \to Y$ be a morphism of quasiprojective sets $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$.*

(i) The map f is continuous under the Zariski topology on X and Y. In particular, the fiber $f^{-1}(q)$ over any point $q \in Y$ is closed in X.

(ii) The graph map $\gamma_f : X \to \mathbb{P}^n \times \mathbb{P}^m$ (defined by $\gamma_f(p) = (p, f(p))$) is regular and induces an isomorphism between X and its image Γ_f (called the graph of f), which is a closed set in $X \times Y$ (and therefore quasiprojective in $\mathbb{P}^n \times \mathbb{P}^m$).

(iii) There exist a projective set X', a morphism $\overline{f} : X' \to \overline{Y}$ ($\overline{Y}$ being the Zariski closure of Y), an open subset U of X' and an isomorphism $\gamma : X \to U$ of quasiprojective sets such that $f = \overline{f}\gamma$.

Proof: Let W be a closed subset of Y. In order to prove that $f^{-1}(W)$ is also closed in X, take a point $p \notin f^{-1}(W)$. We know then from the definition of morphism that in a neighborhood U of p in X the map f is defined by homogeneous polynomials $F_0 \dots, F_m \in \mathbb{K}[X_0, \dots, X_n]$. Let $G_1, \dots, G_s \in \mathbb{K}[Y_0, \dots, Y_m]$ be now a set of generators of $I(W)$ (which coincides with $I(\overline{W})$). Then clearly $U \cap f^{-1}(W)$ is defined in U by the polynomials $G_1(F_0, \dots, F_m), \dots, G_s(F_0, \dots, F_m) \in \mathbb{K}[X_0, \dots, X_n]$. Therefore $U \cap f^{-1}(W)$ is closed in U, and thus we can find a neighborhood V of p in U (and hence also a neighborhood in X) that does not meet $f^{-1}(W)$. This shows that $f^{-1}(W)$ is closed, and hence proves (i).

In order to prove (ii), we first check that the γ_f is regular. Indeed, any point of X has by definition a neighborhood U in which f is defined by homogeneous polynomials $F_0, \dots, F_m \in \mathbb{K}[X_0, \dots, X_n]$ of the same degree. Therefore, in the same neighborhood U, γ_f is defined, after the Segre embedding, by the products X_iF_j, which are all homogeneous of the same degree (and not vanishing simultaneously at any point of U).

Let us see now that Γ_f is closed in $X \times Y$. We take then $(p, q) \notin \Gamma_f$ (i.e. $f(p) \neq q$). Choose a neighborhood $U \subset \mathbb{P}^n$ of p such that f is defined in $U \cap X$ by homogeneous polynomials $F_0, \dots, F_m \in \mathbb{K}[X_0, \dots, X_n]$ of the same degree, say d. Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be the set defined by the 2×2 minors of the matrix

$$\begin{pmatrix} F_0 & \dots & F_m \\ Y_0 & \dots & Y_m \end{pmatrix}.$$

Since these minors are bihomogeneous of bidegree $(d, 1)$ in $\mathbb{K}[X_0, \dots, X_n, Y_0, \dots, Y_m]$, Z is a projective set, so that the intersection of $U \times \mathbb{P}^m$ and the complement of Z in $\mathbb{P}^n \times \mathbb{P}^m$ clearly defines on $X \times Y$ an open neighborhood of (p, q) not meeting Γ_f. Hence the graph is a closed set in $X \times Y$. Since $X \times Y$ is quasiprojective in $\mathbb{P}^n \times \mathbb{P}^m$, then also the graph is quasiprojective.

Finally, it is clear that γ_f and the first projection are inverse to each other when restricted to X and Γ_f, which concludes the proof of (ii).

And for (iii) it is enough to take as X' the Zariski closure of Γ_f, since the projection to $\mathbb{P}^m$ is globally defined, and of course the image of this closure is contained in $\overline{Y}$. This completes the proof of the lemma. □

Example 7.9. Let us see how the proof of the above lemma works in practice. So consider the linear projection of $\mathbb{P}^2$ from the point $p = (0:0:1)$. In equations, we have $f: \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$ defined by $f(x_0 : x_1 : x_2) = (x_0 : x_1)$. Hence the graph $\Gamma_f \subset \mathbb{P}^2 \times \mathbb{P}^1$ is given by the pairs $((x_0 : x_1 : x_2), (y_0 : y_1))$ such that $(x_0, x_1) \neq (0,0)$ and $(x_0 : x_1) = (y_0 : y_1)$, the latter condition being equivalent to $x_0 y_1 = x_1 y_0$. Therefore Γ_f is an open set of the projective set $B = V(X_0Y_1 - X_1Y_0)$, and via the isomorphism $\mathbb{P}^2 \setminus \{p\} \cong \Gamma_f$ we can extend f to the second projection map $B \to \mathbb{P}^1$. Observe that the fiber by the first projection of a point $(x_0 : x_1 : x_2) \neq p$ is the point $((x_0 : x_1 : x_2), (x_0 : x_1))$, while the fiber of $p = (0:0:1)$ is the set $E = \{p\} \times \mathbb{P}^1$ (which becomes a line after the Segre embedding). Interpreting as in Exercise 7.7 $\mathbb{P}^1$ with the pencil of lines through p we see that then the fiber of each point $q \neq p$ is identified with the line $< p, q >$, while E is naturally identified with the pencil of lines through p. In other words, we have replaced p with the set of directions through it (recall that we had an isomorphism $B \setminus E = \Gamma_f \cong \mathbb{P}^2 \setminus \{p\}$). The projective set B is called the *blow-up of* $\mathbb{P}^2$ *along* p and E is called the *exceptional divisor*. It turns out that any regular map can be completed to a morphism between projective varieties in a similar way, i.e. by a blow-up along the indeterminacy locus (see for instance [H], Example 7.18).

Exercise 7.10. Prove that, via the Segre embedding, B is contained in a hyperplane of $\mathbb{P}^5$, so that it can be considered as a surface in $\mathbb{P}^4$. Prove that any line of $\mathbb{P}^2$ passing through p gives rise to a line inside B meeting E, while any line of $\mathbb{P}^2$ not passing through p gives rise to a nondegenerate conic whose linear span does not meet E. Conclude that B can then be described in the following way: Fix in $\mathbb{P}^4$ a line E and a nondegenerate conic C in a plane skew with E, and fix also an isomorphism between E and B; then B consists of the union of the lines joining a point of E with its corresponding point of C.

From part (iii) Lemma 7.8 and what we have remarked in the above example, it is seems enough to study maps between projective sets. In fact, morphisms of projective sets have the following nice and important property.

Theorem 7.11. *Let $f: X \to Y$ be a regular map of projective sets $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$. Then, for any projective set $Z \subset X$, $f(Z)$ is a projective set in $\mathbb{P}^m$ (i.e. f is a closed map for the Zariski topology).*

Proof: Since Z is isomorphic to the graph of $f_{|Z}$, which is closed in $\mathbb{P}^n \times \mathbb{P}^m$ (both things by Lemma 7.8), it is enough to prove the theorem in the case in which $X = \mathbb{P}^n \times \mathbb{P}^m$, $Y = \mathbb{P}^m$ and f is the second projection.

So let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be a closed set defined by bihomogeneous polynomials $F_1, \dots, F_s \in \mathbb{K}[X_0, \dots, X_n, Y_0, \dots, Y_m]$. The image of Z will consist of points $y = (y_0 : \dots : y_m) \in \mathbb{P}^m$ for which the homogeneous polynomials $F_1(X_0, \dots, X_n, y), \dots, F_s(X_0, \dots, X_n, y)$ define a non-empty projective set in $\mathbb{P}^n$. By the weak Nullstellensatz (Theorem 1.24), this is equivalent to the fact that no power of the ideal $\mathfrak{M} = (X_0, \dots, X_n)$ is contained in the ideal I_y generated by those polynomials. Therefore $f(Z)$ will be the intersection of the sets $W_d := \{y \in \mathbb{P}^m \mid \mathfrak{M}^d \not\subset I_y\}$. Since an arbitrary intersection of closed sets is closed, it is then enough to prove that each W_d is closed.

Write $F_i = \sum G_{i;i_0,\dots,i_n} X_0^{i_0} \dots X_n^{i_n}$ (with $G_{i;i_0,\dots,i_n} \in \mathbb{K}[Y_0, \dots, Y_m]$) and let d_i be the degree in $X_0, \dots, X_n$ of the polynomial F_i. The condition $y \in W_d$ is equivalent to say that the elements of $T = \{X_0^{j_0} \dots X_n^{j_n} F_i(X_0, \dots, X_n, y) \mid i = 1, \dots, s \text{ and } j_0 + \dots + j_n = d - d_i\}$ do not form a basis for the vector space S_d of homogeneous polynomials of degree d. The coordinates of those elements with respect to the base given by all the monomials of degree d consist of (many) zeros and some numbers of the type $G_{i;i_0,\dots,i_n}(y_0, \dots, y_n)$. The condition that T is not a basis is equivalent to the vanishing of the maximal minors of the matrix formed by the coordinates of the elements of T. Hence this condition can be expressed by homogeneous polynomials of $\mathbb{K}[Y_0, \dots, Y_m]$ (namely polynomial expressions in the $G_{i;i_0,\dots,i_n}$'s). Therefore W_d is closed and the theorem is proved. $\square$

Remark 7.12. The above result is clearly false if we work in the category of affine sets. The standard example is to take the affine hyperbola $XY = 1$ and project onto the x-axis. Then the image is the whole affine line minus the point 0, which is not an affine set. The situation can be even more complicated than what this example seems to suggest. Take for instance the blow-up B of $\mathbb{P}^2$ along the point $p = (0 : 0 : 1)$ (see Example 7.9) and consider the projective set $L = V(X_0, Y_0) \subset B$. Now restrict the first projection map to $B \setminus L$. Its image is then $\mathbb{P}^2 \setminus \big(V(X_0) \setminus \{p\}\big)$, i.e. an open set of $\mathbb{P}^2$ plus a point in its closure. In general, one defines a *constructible set* as a set obtained as a projective set, minus a projective set, to which previously one has removed a projective set, to which in turn one has removed... (for instance the image we just have obtained is a constructible set). It can be proved that, if one wants to work not only with projective sets but with a wider category (including for instance quasiprojective sets or affine sets), the right category is the category of constructible sets. In this category, the image of a constructible is always constructible (see for instance [H], Theorem 3.16).

Remark 7.13. If you think that the situation cannot become worse than what we have just described, consider for a while what would happen if you drop the condition that your field is algebraically closed. So take for instance the real numbers as the ground field and consider now the example of the apparently innocent circumference $X^2 + Y^2 = 1$. When you project it to the x-axis what you get is the closed interval $[-1, 1]$, which you cannot define by just using equalities. You need in fact to use also inequalities, so the you naturally arrives to the category of *semialgebraic sets* (naively those defined by equalities and inequalities of polynomials). It is a hard theorem (which requires the theorem of elimination of quantifiers in logic) that the image of any semialgebraic set is a semialgebraic set. The reader interested in this theory can take a look at [BCR].

Exercise 7.14. Prove that any regular function $f : X \to \mathbb{K}$ over a projective variety X, where we identify $\mathbb{K}$ as the affine line consisting of $\mathbb{P}^1$ minus one point, is necessarily constant (this is a kind of generalization of Liouville's theorem over the complex numbers).

Example 7.15. One of the main applications of Theorem 7.11 is to use incidence diagrams to prove that several sets obtained by eliminating quantifiers are projective. For instance, if we consider inside $\mathbb{P}^{\binom{n+d}{d}-1}$ (the projective space of hypersurfaces of degree d in $\mathbb{P}^n$) the set Z of those hypersurfaces containing a k-plane, this is a projective set. Indeed, we know from Exercise 6.5 that the subset of $\mathbb{P}^{\binom{n+d}{d}-1} \times \mathbb{G}(k, n)$ of pairs (Λ, X) for which $\Lambda \subset X$ is a closed set. Hence, its image in $\mathbb{P}^{\binom{n+d}{d}-1}$ under the first projection is also a closed set. But this image is nothing but Z. We will see also later how to get estimates of dimensions using incidence varieties.

Exercise 7.16. Prove that the set of cones is a closed set inside the projective space of hypersurfaces of degree d in $\mathbb{P}^n$. Idem for the set of reducible surfaces.

Example 7.17. Another application of both Lemma 7.8(iii) and Theorem 7.11 is that they allow to define projective sets just on dense subsets. To see a concrete example, let $X \subset \mathbb{P}^n$ be a projective set. If $\Delta \subset X \times X$ is the diagonal, then the map $X \times X \backslash \Delta \to \mathbb{G}(1, n)$ associating to each pair of points is regular (we leave the proof as an exercise). We know then that it can be extended (after some isomorphism) to a regular map $\tilde{Z} \to \mathbb{G}(1, k)$. The image is then a projective set $SX \subset \mathbb{G}(1, n)$, which contains as a dense subset the image of the original map.

We prove now some first properties of morphisms among projective sets.

Proposition 7.18. *Let $f : X \to Y$ be a morphism between projective sets $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$.*

(i) If Z is an irreducible closed subset of X, then also $f(Z)$ is irreducible.

(ii) For any closed subset $Z \subset X$, $\dim f(Z) \leq \dim Z$. In particular, isomorphisms preserve dimensions.

(iii) For any $k \in \mathbb{Z}$, the set Z_k of points of Y whose fiber has dimension at least k is a projective set.

Proof: Since the restriction of a morphism to any projective set is also a morphism, it is clear that it is enough to prove (i) and (ii) in the case $Z = X$. So assume first that $f(X)$ (which is closed by Theorem 7.11) is a nontrivial union of two projective sets W_1 and W_2. Then X will also be the nontrivial union of $f^{-1}(W_1)$ and $f^{-1}(W_2)$, which are closed by Lemma 7.8(i). This contradicts the assumption that X is irreducible and proves that $f(X)$ is irreducible, i.e. (i).

To prove (ii), write $r = \dim f(Z)$. Take $Z'_r \subset f(Z)$ an irreducible component of $f(Z)$ of dimension r. It could happen that $f^{-1}(Z'_r)$ is not irreducible, but its image is Z'_r, which is irreducible. Therefore there exists an irreducible component Z_r of $f^{-1}(Z'r)$ such that $f(Z_r) = Z'_r$. We restrict our attention to the restriction $f_{|Z_r} : Z_r \to Z'_r$. Take now Z'_{r-1} an irreducible set of Z'_r of dimension $r-1$. By the same reason as above, there is a component Z_{r-1} of $f^{-1}_{|Z_r}(Z'_{r-1})$ mapping onto Z'_{r-1} (and in particular $Z_{r-1} \subsetneq Z_r$). Iterating the process, we find a chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_r$ of irreducible sets of Z, so that $\dim Z \geq r$, which proves (ii).

For part (iii) recall from Proposition 5.7(vi) that $f^{-1}(y)$ has dimension at least k if and only if it meets all linear subspaces of $\mathbb{P}^n$ of codimension k. Hence Z_k is the intersection of all the sets $Z_\Lambda = f(X \cap \Lambda)$, as Λ varies in $\mathbb{G}(n-k, n)$. Since each Z_Λ is closed by Theorem 7.11, (iii) follows immediately. □

Exercise 7.19. Given a projective set $X \subset \mathbb{P}^n$, prove that the set of linear spaces of dimension k meeting X in a set of dimension at least r is a projective set of $\mathbb{G}(k, n)$ for each $r \in \mathbb{N}$.

We prove now a lemma, which will be crucial to prove the important Theorem 7.21 and will be also used in Chapter 8. It gives a very intuitive notion of dimension: it is the dimension of a projective space to which it is possible to map our variety in such a way that all fibers are finite.

Lemma 7.20. *Let $X \subset \mathbb{P}^n_{\mathbb{K}}$ be a projective set of dimension r and let $F_0, \ldots, F_r \in \mathbb{K}[X_0, \ldots, X_n]$ be homogeneous polynomials of the same degree d such that X does not meet $V(F_0, \ldots, F_r)$.*

(i) The morphism $f : X \to \mathbb{P}^r_{\mathbb{K}}$ defined by

$$f(a_0 : \ldots : a_n) = (F_0(a_0, \ldots, a_n) : \ldots : F_r(a_0, \ldots, a_n))$$

is surjective.

(ii) *For any $y \in \mathbb{P}^r$ the fiber $f^{-1}(y)$ is a finite number of points.*

(iii) *If X is irreducible, for any homogeneous polynomial $G \in \mathbb{K}[X_0, \dots, X_n]$ of degree d, the image of the map $f' : X \to \mathbb{P}^{r+1}_{\mathbb{K}}$ defined by*

$$f'(a_0 : \ldots : a_n) = (F_0(a_0, \dots, a_n) : \ldots : F_r(a_0, \dots, a_n) : G(a_0, \dots, X_n))$$

is an irreducible hypersurface in $\mathbb{P}^{r+1}_{\mathbb{K}}$ not containing the point $(0 : \ldots : 0 : 1)$.

Proof: Let $y \in \mathbb{P}^r$ any point of $\mathbb{P}^r$. After a linear change of coordinates (which will change $F_0, \dots, F_r$ in other polynomials of degree d with the same property of defining a projective set not meeting X), we can assume that $y = (1 : 0 : \ldots : 0)$. Then $f^{-1}(y) = V(F_1, \dots, F_r) \cap X$, which is not empty by Proposition 5.7(viii)). This proves (i). But $f^{-1}(y)$ cannot have dimension at least one, since this would imply, by the same reason, that $V(F_1, \dots, F_r) \cap X$ should meet $V(F_0)$, contrary to our hypothesis. Hence (ii) holds.

Part (iii) is analogous. Indeed we know that $f'(X)$ is irreducible and projective, and its dimension is at most r. Since the inverse image of a line (i.e. the intersection of r hyperplanes of $\mathbb{P}^{r+1}_{\mathbb{K}}$) is the intersection of X with r hypersurfaces of $\mathbb{P}^n_{\mathbb{K}}$ of degree d, it follows that it is not empty. Hence any line meets the image of f', hence $f'(X)$ has dimension r, and being irreducible it is an irreducible hypersurface (see Proposition 5.9). The hypothesis $X \cap V(F_0, \dots, F_r) = \emptyset$ implies that the point $(0 : \ldots : 0 : 1)$ is not in the image of f', as wanted. □

The following result is a much stronger version of Proposition 5.7(ii). It will imply that many statements about the dimension of projective sets can in fact be generalized to any of the components of the projective set itself. This will allow us to work often just locally instead of with the whole projective set we will be dealing with (for instance when we will study the tangent space at a point).

Theorem 7.21. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension r and let $V(F)$ be a hypersurface not containing X. Then any irreducible component of $X \cap V(F)$ has dimension $r-1$. As a corollary, the components of the intersection of X with s hypersurfaces will all have dimension at least $r-s$*

Proof: Let $X \cap V(F) = X_1 \cup \ldots \cup X_s$ be the decomposition into irreducible components. Assume for contradiction that for instance $\dim(X_s) \leq r-2$. Since $X_s \not\subset X_1 \cup \ldots \cup X_{s-1}$, then $I(X_1) \cap \ldots \cap I(X_{s-1}) \not\subset I(X_s)$. We thus can find a homogeneous polynomial $G \in I(X_1) \cap \ldots \cap I(X_{s-1}) \setminus I(X_s)$. After replacing F with $F^{\deg(G)}$ and G with $G^{\deg(F)}$, we can assume that F and G has the same degree. We now take a family $F_0, \dots, F_r = F$ of homogeneous polynomials of degree d such that $X \cap V(F_0, \dots, F_r) = \emptyset$. We construct the following morphisms:

–The map $X_s \to \mathbb{P}^{r-1}$ defined by $(F_0 : \ldots : F_{r-1})$, which cannot be surjective because $\dim(X_s) \leq r-2$. If $P \in \mathbb{K}[T_0, \ldots, T_{r-1}]$ is a nonzero homogeneous polynomial in the ideal of the image, then it follows $P(F_0, \ldots, F_{r-1}) \in I(X_s)$. Therefore $P(F_0, \ldots, F_{r-1})G \in I(X \cap V(F_r))$, hence there exists m such that $P(F_0, \ldots, F_{r-1})^m G^m \in I(X) + (F_r)$. We can thus write

$$P(F_0, \ldots, F_{r-1})^m G^m \equiv HF_r \pmod{I(X)} \tag{$*$}$$

for some homogeneous polynomial H of degree say l.

–The map $X \to \mathbb{P}^{r+1}$ defined by $(F_0^l : \ldots : F_r^l : H^d)$, whose image, by Lemma 7.20, is an irreducible hypersurface not containing $(0 : \ldots : 0 : 1)$.Let $Q \in \mathbb{K}[T_0, \ldots, T_{r+1}]$ be the equation of that hypersurface, which we write as a polynomial in T_{r+1}:

$$Q = A_e + A_{e-1}T_{r+1} + \ldots + A_1 T_{r+1}^{e-1} + A_0 T_{r+1}^e$$

where each A_i is a homogeneous polynomial of degree i in $\mathbb{K}[T_0, \ldots, T_r]$, and $A_0 \neq 0$. We thus have

$$A_e(F_0^l, \ldots, F_r^l) + A_{e-1}(F_0^l, \ldots, F_r^l)H^d + \ldots + A_1(F_0^l, \ldots, F_r^l)H^{d(e-1)} + A_0 H^{de} \in I(X).$$

Multiplying by F_r^{de} and taking congruences modulo $I(X)$ (see (*)) we get

$$A_e(F_0^l, \ldots, F_r^l)F_r^{de} + A_{e-1}(F_0^l, \ldots, F_r^l)F_r^{d(e-1)}P(F_0, \ldots, F_{r-1})^{md}G^{md} + \ldots +$$

$$+A_1(F_0^l, \ldots, F_r^l)F_r^d P(F_0, \ldots, F_{r-1})^{md(e-1)}G^{md(e-1)} + A_0 P(F_0, \ldots, F_{r-1})^{mde}G^{mde} \equiv 0.$$

This means that the polynomial

$$R := A_e(T_0^l, \ldots, T_r^l)T_r^{de} + A_{e-1}(T_0^l, \ldots, T_r^l)T_r^{d(e-1)}P(T_0, \ldots, T_{r-1})^{md}T_{r+1}^{md} + \ldots +$$

$$+A_1(T_0^l, \ldots, T_r^l)T_r^d P(T_0, \ldots, T_{r-1})^{md(e-1)}T_{r+1}^{md(e-1)} + A_0 P(T_0, \ldots, T_{r-1})^{mde}T_{r+1}^{mde}$$

is in the ideal of the image of the map $X \to \mathbb{P}^{r+1}$ defined by $(F_0 : \ldots : F_r : G)$. By Lemma 7.20, that ideal is generated by an irreducible polynomial R' that is monic in the variable T_{r+1}, i.e. R is divisible by R'. Since modulo (T_r) the polynomial R is the nonzero monomial $A_0 P(T_0, \ldots, T_{r-1})^{mde}T_{r+1}^{mde}$, this means that modulo (T_r) the polynomial R' is also a monomial of the form T_{r+1}^c. Therefore R' takes the form

$$R' = B_{c-1}(T_0, \ldots, T_r)T_r + B_{c-2}(T_0, \ldots, T_r)T_r T_{r+1} + \ldots + B_0(T_0, \ldots, T_r)T_r T_{r+1}^{c-1} + T_{r+1}^c$$

with each B_i homogeneous of degree i. This implies that

$$B_{c-1}(F_0, \ldots, F_r)F_r + B_{c-2}(F_0, \ldots, F_r)F_r G + \ldots + B_0(F_0, \ldots, F_r)F_r G^{c-1} + G^c \in I(X)$$

so that G is in the radical of $I(X)+(F_r)$, which is the ideal of $X \cap V(F_r)$, contrary to the assumption $G \notin I(X_s)$. □

Remark 7.22. Observe that the above theorem does not imply that all the primary components of $I(X)+(F)$ has dimension $r-1$. In fact Exercise 2.13 shows that this is not true. If this example does not convince you because the extra component has dimension -1, just take the same equations in $\mathbb{K}[X_0, X_1, X_2, X_3, X_4]$ (i.e. consider a cone over the curve X) to obtain a surface in $\mathbb{P}^4$ whose intersection with a hyperplane has an embedding component (corresponding to the point $(0:0:0:0:1)$, the vertex of the cone).

Remark 7.23. The proof of Theorem 7.21 can be adapted to prove that, if $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is irreducible of dimension r and F is a bihomogeneous outside $\overline{I}(X)$, then any irreducible component of $X \cap V(F)$ has dimension $r-1$. If F has bidegree (a,a) then the result is in fact an immediate consequence of Theorem 7.21, since then we can regard $X \cap V(F)$ as the intersection in $\mathbb{P}^{nm+n+m}$ of the image of X under the Segre embedding and a hypersurface $V(G)$, where G is a homogeneous polynomial of degree a whose restriction to $\mathbb{P}^n \times \mathbb{P}^m$ coincides with F. If instead F has bidegree (a,b) for instance with $a<b$, we can also conclude by observing that any component of $V(F)$ is a components of the intersection of X with some $V(X_i^{b-a}F)$, $i=0,\dots,n$. We will see in Proposition 10.9 a much stronger generalization of Theorem 7.21.

8. Properties of morphisms

We start this chapter by proving that the local definition of regular map can be extended to a global definition when the source is the whole projective space.

Proposition 8.1. *Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a regular map. Then there exist homogeneous polynomials $F_0, \dots, F_m \in \mathbb{K}[X_0, \dots, X_n]$ of the same degree such that $f(x_0, \dots, x_n) = (F_0(x_0, \dots, x_n) : \dots : F_m(x_0, \dots, x_n))$ for any $(x_0 : \dots : x_n) \in \mathbb{P}^n$.*

Proof: From the definition of regular map, for any $x = (x_0 : \dots : x_n)$ we can find $F_0, \dots, F_m$ defining f in an open neighborhood of x. Dividing by the greatest common divisor, we can assume that $F_0, \dots, F_m$ have no common irreducible factors. We take now two arbitrary points $x, x' \in \mathbb{P}^n$, and want to prove that the corresponding (local) homogeneous polynomials are proportional, in the sense that there exists $\lambda \in \mathbb{K}$ such that $F_i' = \lambda F_i$ for each $i = 0, \dots, m$. We first observe that the matrix

$$\begin{pmatrix} F_0 & \dots & F_m \\ F_0' & \dots & F_m' \end{pmatrix}$$

has rank one at any point y in the (non-empty) intersection of the open neighborhoods of x and x'. Therefore all the minors $F_i F_j' - F_j F_i'$ are zero in that open set, and hence they are identically zero. In order to be able to use linear algebra, we have to regard the homogeneous polynomials as elements of the field $K = \{\frac{A}{B} \mid A \text{ and } B \text{ are homogeneous polynomials}\}$. We thus now that there exists $\frac{A}{B} \in K$ (we can assume A and B to be coprime) such that $AF_i' = BF_i$ for each $i = 0, \dots, m$. But since no factor of A (resp. B) can divide neither B (resp. A) nor all the F_i's (resp. F_i''s) it follows that A and B are constants, just proving the proposition. □

Exercise 8.2. Prove that any regular map $f : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^r$ is globally defined by $r + 1$ bihomogeneous polynomials of the same bidegree.

The same proof as in Proposition 8.1 yields a stronger statement in case $n = 1$.

Proposition 8.3. *Let $f : U \to \mathbb{P}^n$ be a regular map defined on an open set U of $\mathbb{P}^1$. Then there exists a morphism $\mathbb{P}^1 \to \mathbb{P}^n$ such that its restriction to U coincides with f.*

Proof: Let $F_0, \dots, F_n \in \mathbb{K}[X_0, X_1]$ be homogeneous polynomials of the same degree, with no common factors, defining f in some open set, maybe smaller than U. The difference now with the proof of Proposition 8.1 is that the fact that $F_0, \dots, F_n$ has no common factors is equivalent to say that there is no $(a_0 : a_1) \in \mathbb{P}^1$ vanishing at all of them (for otherwise $a_1 X_0 - a_0 X_1$ would be a common factor). Hence we can repeat the proof of Proposition 8.1 to conclude that the map defined by $F_0, \dots, F_n$ extends f. □

It seems natural that the dimension statements of Proposition 7.18 could be improved in the sense that the dimension of the source of a (surjective) morphism must be the sum of the dimension of the target plus the dimension of a general fiber. This is what we will prove in the next theorem.

Theorem 8.4. *Let $f : X \to Y$ be a surjective morphism between projective sets $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$.*

(i) For any $y \in Y$, the fiber $f^{-1}(y)$ has dimension at least $\dim X - \dim Y$. Moreover, if X is irreducible, all the components of any $f^{-1}(y)$ have dimension at least $\dim X - \dim Y$.

(ii) For $k = \dim X - \dim Y$, the open set $Y \setminus Z_{k+1}$ of points of Y with k-dimensional fiber is not empty (in other words, the dimension of X is the dimension of the image Y plus the dimension of a general fiber).

(iii) If Y is irreducible and all the fibers of f are irreducible of the same dimension c, then X is irreducible of dimension $c + \dim Y$.

Proof: Clearly it is enough to prove (i) and (ii) when X is irreducible. Since from Lemma 7.20 and Proposition 5.7(vi) we have a morphism $g : Y \to \mathbb{P}^m$ (where $m = \dim Y$) having finite fibers, it is enough to prove parts (i) and (ii) when Y is a projective space. Indeed, given $z \in \mathbb{P}^s$, if $g^{-1}(z) = \{y_1, \dots, y_k\}$, then $(gf)^{-1}(z) = f^{-1}(y_1) \cup \dots \cup f^{-1}(y_k)$. Therefore, since the above union is disjoint, the irreducible decomposition of $(gf)^{-1}(z)$ is the union of the irreducible decompositions of $f^{-1}(y_1), \dots, f^{-1}(y_k)$. Hence irreducible components of fibers of f are irreducible components of fibers of gf and viceversa. And clearly as an open set satisfying the statement in (ii) it is enough to take the inverse image by g of the open set found for $\mathbb{P}^m$.

So if $f : X \to \mathbb{P}^m$ is a surjective morphism, the fiber over a point $y \in \mathbb{P}^m$ is isomorphic to the intersection of the graph $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^m$ with $\mathbb{P}^n \times \{y\}$. But the latter is just obtained as the intersection of m bihomogeneous forms in $\mathbb{K}[X_0, \dots, X_n, Y_0, \dots, Y_m]$ of bidegree $(0, 1)$. Hence part (i) and (ii) comes by Proposition 6.11 and Remark 7.23. Indeed, if the bihomogeneous forms are good enough (it is enough to take them in an iterative way not containing any irreducible component of the previous intersection) then their intersection will define a point $y \in \mathbb{P}^m$ such that all the components of its fiber have dimension exactly $\dim X - m$. Hence the set $Y \setminus Z_{k+1}$ is not empty, proving (ii). On the other hand, if at some step the intersection is not good, the dimension of some component could remain the same, hence we have that, for any $y \in \mathbb{P}^m$, all the components of $f^{-1}(y)$ have dimension at least $\dim X - m$, proving (i).

In order to prove (iii) we go back to the situation in which Y is not necessarily $\mathbb{P}^m$, but making now the assumption that it is irreducible. For any irreducible component Z_i of X, consider either $Y \setminus f(Z_i)$ if $f_{|Z_i}$ is not surjective or otherwise the open subset $V_i \subset Y$ on which $\dim(f^{-1}(y)\cap Z_i)=\dim Z_i-\dim Y$ (which is not empty by part (ii)). Take $y_0\in\bigcap_i V_i$. Since by assumption $f^{-1}(y_0)$ is irreducible of dimension c, it follows that $f^{-1}(y_0)$ (which is the union of the fibers of all the $f_{|Z_i}$'s) coincides with some $f^{-1}(y_0)\cap Z_{i_0}$. In particular, $f_{|Z_{i_0}}$ is surjective and $c=\dim Z_{i_0}-\dim Y$. Hence from part (i) $\dim(f^{-1}(y)\cap Z_{i_0})\geq c$ for any $y\in Y$. But since $f^{-1}(y)$ is irreducible of dimension c, it follows immediately that $f^{-1}(y)=f^{-1}(y)\cap Z_{i_0}$ for any $y\in Y$. In other words, all the fibers of f are contained in Z_{i_0}, which implies that Z_{i_0} is the only irreducible component of X, and part (iii) is then proved. $\square$

Exercise 8.5. Prove that $\mathbb{G}(k,n)$ is irreducible of dimension $k(n-k)$ [Hint: some induction argument could be very useful].

Exercise 8.6. For any $k,n\in\mathbb{N}$, find a bound $d(k,n)$ such that for any $d\geq d(k,n)$ there exist hypersurfaces of degree d in $\mathbb{P}^n$ not containing any linear space of dimension k. For $n=3,k=1$ find a sharp bound [Hint: it could be useful to prove that $V(X_0^3+X_1^3+X_2^3+X_3^3)$ contains a finite number of lines, namely 27].

Exercise 8.7. Prove that the projective set of matrices $(n+1)\times(m+1)$ of rank at most k (see Example 1.7) is irreducible and has codimension $(n+1-k)(m+1-k)$. Find the corresponding codimension when the matrices are symmetric (Example 1.7) or skew-symmetric (Example 1.15).

Exercise 8.8. Prove that the set of k-planes in $\mathbb{P}^n$ meeting a fixed linear space of dimension r in a linear space of dimension at least s (with $r+k-n\leq s\leq\min\{k,r\}$) is a projective variety of codimension $(s+1)(n-k-r+s)$ in $\mathbb{G}(k,n)$.

Exercise 8.9. Prove that, for $s=0$, the result in the above exercise remains true when replacing the linear space of dimension r by an arbitrary variety of dimension r.

We want to generalize now Theorem 8.4 proving that not only the dimension of the fiber is constant in an open set of the target, but even the whole Hilbert polynomial. For this we will need to fix some general set-up.

Definition. The *rational function field $K(X)$ of a projective variety $X\subset\mathbb{P}^n$* is the field consisting of the quotients of homogeneous elements of $S(X)$ with the same degree.

If now we have a surjective morphism $f:X\to Y$ between two projective sets, after replacing X with its graph we can assume that X is a projective set inside $\mathbb{P}^n\times\mathbb{P}^m$ and that f is the restriction of the projection onto the second factor. We have then a natural inclusion $S(Y)\hookrightarrow\overline{S}(X)$. If Y is irreducible, we can consider the (classes of) polynomials in $\mathbb{K}[X_0,\dots,X_n,Y_0,\dots,Y_m]$ as (classes of) polynomials in $K(Y)[X_0,\dots,X_n]$. We can hence extend $\overline{S}(X)$ to a quotient S_f of the polynomial ring $K(Y)[X_0,\dots,X_n]$ by the ideal I_f generated by the polynomials in $\overline{I}(X)$.

Definition. We will call the *Hilbert function of the morphism* f to the Hilbert function h_f of I_f, and the *Hilbert polynomial of* f to the Hilbert polynomial P_f of I_f (recall from Remark 3.14 or Chapter 4 that the Hilbert polynomial exists even if the field $K(Y)$ is not algebraically closed).

The main result is that the Hilbert function and polynomial of a regular map are exactly the ones of a general fiber. We need however to be careful about our definition of fiber.

Definition. The *fiber ideal of a surjective morphism* $f : X \to Y$ *at a point* $y \in Y$ is the ideal I_y obtained from I_f by substituting the variables $Y_0,\dots,Y_n$ by the coordinates of y. Observe that $V(I_y)$ coincides with $f^{-1}(y)$ (as subsets in $\mathbb{P}^n$), but it is not true in general that $I(f^{-1}(y)) = I_y$.

Example 8.10. Let us give an example to clarify the above concepts. Consider $f : \mathbb{P}^1 \to \mathbb{P}^1$ the map given by $f(X_0 : X_1) = (X_0^2 : X_1^2)$. Its graph $\Gamma_f \subset \mathbb{P}^1 \times \mathbb{P}^1$ is thus given by $V(X_0^2Y_1 - X_1^2Y_0)$. The ideal $I_f \subset K(\mathbb{P}^1)$ is generated by $X_0^2 - X_1^2\frac{Y_0}{Y_1}$ or $\frac{Y_1}{Y_0}X_0^2 - X_1^2$ (depending on our convenience, we can choose whatever we want, since they are proportional). This generator is irreducible and the Hilbert polynomial P_f of I_f is constant equal to 2. In fact, also the Hilbert polynomial of the ideal $I_y \subset \mathbb{K}[X_0, X_1]$ generated by $F_y = X_0^2y_1 - X_1^2y_0$ is two for any $y = (y_0 : y_1) \in \mathbb{P}^1$. But for $y = (1 : 0)$ or $y = (0 : 1)$ (and only for these values), the polynomial F_y has a double root. This is just saying that the fiber of f consists of two points for any $y \in \mathbb{P}^1$, the two points being infinitely close for those two particular values.

Theorem 8.11. *Let* $f : X \to Y$ *be a surjective morphism of projective sets, and assume* Y *is irreducible.*

(i) For any $l >> 0$*, there exists an open set* $V_l \subset Y$ *such that for any* $y \in V_l$ *it holds* $h_y(l) = h_f(l)$*.*

(ii) There exists an open set $V \subset Y$ *such that the Hilbert polynomial of* I_y *is* P_f *for any* $y \in V$*.*

Proof: Fix $l \in \mathbb{N}$, and choose monomials $M_1,\dots,M_s$ of degree l in $X_0,\dots,X_n$ such that their classes modulo I_f form a basis of $(S_f)_l$ over $K(Y)$ (of course we can do so, since the monomials of degree l generate $(S_f)_l$). To prove (i) we will see that the classes of $M_1,\dots,M_s$ modulo I_y will be a basis of $(S/I_y)_l$ for y in a suitable open set $V_l \subset Y$.

We will see first under which conditions the classes of $M_1,\dots,M_s$ generate $(S/I_y)_l$. We first observe that any monomial M of degree l can be written as $G_1M_1 + \dots + G_sM_s +$

$H_1F_1+\ldots+H_tF_t$, where $G_1,\ldots,G_s \in K(Y)$, $H_1,\ldots,H_\in K(Y)[X_0,\ldots,X_n]$ and $F_1,\ldots,F_s$ is a system of generators of $\overline{I}(X)$. For the values of y in the open set V_M consisting of the points not vanishing at the denominators of $G_1,\ldots,G_s,H_1,\ldots,H_t$, it also holds that M can be written modulo I_y as a linear combination of $M_1,\ldots,M_s$ with coefficients in $\mathbb{K}$. The first open set V_l' we are looking for is just take the (finite) intersection of the open sets V_M.

To complete the proof of (i) we have to check the linear independency of the classes of $M_1,\ldots,M_s$ modulo I_y over $\mathbb{K}$. The fact that they are linearly independent modulo I_f over $K(Y)$, is equivalent (knowing that they generate) to the fact that the linear subspace of $W \subset K(Y)[X_0,\ldots,X_n]_l$ generated by the products of each F_i with all the monomials of degree $l-\deg F_i$ (we take for this l bigger than the maximum of the degrees of the F_i's) has dimension $\binom{n+l}{n}-s$. This condition can be expressed by vanishing suitable minors of a matrix with entries in $K(Y)$. Taking an open set $V_l \subset V_l'$ in which no denominator of the entries of this matrix vanishes, we get that for each $y \in V_l$ the classes of $M_1,\ldots,M_s$ modulo I_y over $\mathbb{K}$ are also linearly independent.

To prove (ii), we first take l_0 such that for $l \geq l_0$ it holds $h_f(l) = P_f(l)$. We observe that this l_0 can be taken depending on the maximum negative twist appearing in a free resolution of S_f as a graded $K(Y)[X_0,\ldots,X_n]$-module (recall that in Theorem 4.3 the ground field did not need to be algebraically closed). It is not difficult to see (but we will not check the details here) that such a resolution remains exact when specializing for points y in an open set $V' \subset Y$. This means that also $h_y(l) = P_y(l)$ if $l \geq l_0$ and $y \in V'$. Obviously, all these Hilbert polynomials have degree at most n, so that they are determined by their values in $n+1$ points (if you want to be accurate, after shrinking the open set of Y and using Theorem 8.4(ii) the degree is bounded by $\dim X - \dim Y$). It is thus clear that the open set $V = V' \cap \left(\bigcap_{l=l_0}^{l_0+n} V_l\right)$ satisfies the wanted property. This completes the proof. □

Definition. A regular map $f : X \to Y$ is *flat* if the Hilbert polynomial of I_y is P_f for any $y \in Y$.

Example 8.12. From this point of view Example 1.23 should be clearer now. First we homogenize it to obtain a morphism of projective sets. We thus consider $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ defined by the ideal $(X_1X_2, X_1X_3, Y_0X_2X_3 - Y_1X_0X_2, Y_0X_3^2 - Y_1X_0X_3)$ (it can be shown that it is radical). We consider then the map $f : X \to \mathbb{P}^1$ induced by the projection onto the second factor. For any $y \neq (1:0)$, the ideal I_y is the ideal of two skew lines, so with Hilbert polynomial $2l+2$, while for $y = (1:0)$ we get the ideal $I_{(1:0)} = (X_1X_2, X_1X_3, X_2X_3, X_3^2)$, which is not radical, but we observed in Exercise 5.21 that its Hilbert polynomial is also $2l+2$. Therefore f is a flat morphism.

We end this chapter by proving that the arithmetic genus of a projective set is, despite of its definition in terms of the Hilbert polynomial, invariant under isomorphisms. At a first glance it could seem trivial, since, given a morphism $X \to Y$, there are natural isomorphisms between $S(X)_l$ and $\overline{S}(\Gamma_f)_{l,0}$, and between $S(Y)_l$ and $\overline{S}(\Gamma_f)_{0,l}$. However this is not enough to conclude, since we only now that $\overline{P}_{\Gamma_f}(l_1, l_2)$ gives the dimension of $\overline{S}(\Gamma_f)$ for big values of l_1 and l_2, but not necessarily for $l_1 = 0$ or $l_2 = 0$. The key observation will be thus to observe that the graph of f, when intersected with hypersurfaces of bidegree $(0, 1)$ will never produce $\overline{\mathfrak{M}}_2$-primary components (see the Remark 6.9). We will prove first some lemmas.

Lemma 8.13. *Let $f : X \to Y$ be a regular map between to projective sets $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. Assume that there exists an affine open set $D(F) \subset \mathbb{P}^n$ such that $f(X \cap D(F)) \subset D(Y_0)$ (i.e. the image is contained in an affine space). Then there exist homogeneous forms $F_0, \ldots, F_m \in \mathbb{K}[X_0, \ldots, X_n]$ of the same degree such that $f(x_0 : \ldots : x_n) = (F_0(x_0, \ldots, x_n) : \ldots : F_m(x_0, \ldots, x_n))$ for any $(x_0 : \ldots : x_n) \in D(F)$, and moreover F_0 can be taken to be a power of F.*

Proof: For any $x \in X \cap D(F)$ we can find a neighborhood $X \cap D(G)$ of x on which f is defined by homogeneous polynomials $G_0, \ldots, G_m \in \mathbb{K}[X_0, \ldots, X_n]$. Multiplying all these polynomials by G and changing G by GG_0 we can assume that $G = G_0$ and that $G_0(x') = \ldots = G_m(x') = 0$ if $x' \notin D(G)$. We can then cover $X \cap D(F) = \bigcup_i (X \cap D(G_{i0}))$ with open sets of the above form, i.e. such that on $X \cap D(G_{i0})$ the map f is defined by homogeneous polynomials $G_{i0}, \ldots, G_{im}$. The obvious inclusion $X \cap (\bigcap_i V(G_{i0})) \subset V(F)$ implies that F belongs to the homogeneous ideal of $X \cap (\bigcap_i V(G_{i0}))$, which by the Nullstellensatz (Theorem 3.17) is the radical of the sum of $I(X)$ and the ideal generated by the G_{i0}'s. Hence we have an expression of the form $F^d = H + \sum_i H_i G_{i0}$, where the sum is finite, $H \in I(X)$ and the H_i's are homogeneous polynomials. It is then clear that we can take $F_j = \sum_i H_i G_{ij}$ for each $j = 0, \ldots, m$. Indeed, for any $x \in X \cap D(F)$ we have that $H(x) = 0$, and that either $G_{i0}(x) = \ldots = G_{im}(x) = 0$ (if $x \notin D(G_{i0})$) or $f(x) = (G_{i0}(x) : \ldots : G_{im}(x))$ (if $x \in D(G_{i0})$). □

Lemma 8.14. *Let f be a morphism between projective sets $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ and let $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^m$ be its graph. For any $s = 0, \ldots, m$, consider the ideal $J = \overline{I}(\Gamma_f) + (Y_{s+1}, \ldots, Y_m)$*

(i) If $V(J) \neq \emptyset$, then J has at most one $\overline{\mathfrak{M}}_2$-primary component, and necessarily with radical $\overline{\mathfrak{M}}_1 + \overline{\mathfrak{M}}_2$.

(ii) If $V(J) = \emptyset$, then any primary component of J either contains $\overline{\mathfrak{M}}_2$ or is $\overline{\mathfrak{M}}_1$-primary.

Proof: For part (i), assume $\overline{I}(\Gamma_f) + (Y_{s+1}, \ldots, Y_m) = I' \cap I_0$, with I' an $\overline{M}_2$-primary component and I_0 the intersection of the remaining components. Then we can find a bihomogeneous polynomial $F \in I_0 \setminus I'$. We take then an F of bidegree (a,b) with the condition that b is minimum among all the bihomogeneous polynomials not belonging to I' but belonging to all the primary components of $\overline{I}(\Gamma_f) + (Y_{s+1},\ldots,Y_m)$ that are not $\mathfrak{M}_2$-primary. I claim that if $\sqrt{I_0} \neq \mathfrak{M}_1 + \mathfrak{M}_2$ then $b = 0$.

Indeed, assume $b > 0$. Then we can write $F = Y_0G_0 + \ldots + Y_mG_m$ for some homogeneous polynomials. Now the idea is to substitute in the above equality $Y_0,\ldots,Y_m$ with a local representation $F_0,\ldots,F_m$ for the map f, thus getting a polynomial with smaller b. Clearly X is covered by principal sets $D(F')$ as in the statement of Lemma 8.13 (i.e. principal sets whose image by f is contained in some principal set). Therefore by the weak Nullstellensatz (Theorem 1.24) $I(X)$ and those F' generate an $(X_0,\ldots,X_n)$-primary ideal. Since all the polynomials in $I(X)$ are also in $\overline{I}(\Gamma_f)$ (and hence in I') it follows that we can find such an F' not in $\sqrt{I'}$. In other words, in $D(F')$ we have that f is defined by homogeneous polynomials $F'_0,\ldots,F'_m$ with F'_0 being a power of F'. We can also multiply if necessary all these polynomials by F' to conclude that the minors of the matrix

$$\begin{pmatrix} Y_0 & \ldots & Y_m \\ F'_0 & \ldots & F'_m \end{pmatrix}$$

belong to $\overline{I}(\Gamma_f)$. Therefore, modulo $\overline{I}(\Gamma_f)$ (and thus also modulo I') we have that $F'_0F = Y_0F'_0G_0 + \ldots + Y_mF'_0G_m \equiv Y_0(F'_0G_0 + \ldots + F'_mG_m)$. Since $F'_0F \notin I'$ (because I' is primary, $F' \notin \sqrt{I'}$ and $F \notin I'$) it follows that $F'_0G_0 + \ldots + F'_mG_m \notin I'$. In a similar way we then get that $Y_i(F'_0G_0 + \ldots + F'_mG_m) \equiv F'_iF$ modulo $\overline{I}(\Gamma_f)$ for each $i = 0\ldots,m$. This implies that $F'_0G_0 + \ldots + F'_mG_m$ belong to all the primary components of $\overline{I}(\Gamma_f) + (Y_{s+1},\ldots,Y_m)$ that are not $\mathfrak{M}_2$-primary. Since it has bidegree $(a', b-1)$, this is a contradiction, which proves the claim.

We can therefore find a homogeneous polynomial $F \in \mathbb{K}[X_0,\ldots,X_n]$ such that, regarded as a polynomial in $\mathbb{K}[X_0,\ldots,X_n,Y_0,\ldots,Y_m]$, does not belong to I' but belongs to all the primary components of $\overline{I}(\Gamma_f) + (Y_{s+1},\ldots,Y_m)$ that are not $\mathfrak{M}_2$-primary. In particular, $F \notin \overline{I}(\Gamma_f) + (Y_{s+1},\ldots,Y_m)$, but $F \in \sqrt{\overline{I}(\Gamma_f) + (Y_{s+1},\ldots,Y_m)}$. But, as F does not depend on $Y_0,\ldots,Y_m$, it follows that it belongs to $\sqrt{\overline{I}(\Gamma_f)}$, which is a radical ideal. Hence $F \in \overline{I}(\Gamma_f) \subset \overline{I}(\Gamma_f) + (Y_{s+1},\ldots,Y_m) \subset I'$, and we thus get a contradiction. This finishes the proof of (i).

As for (ii), let I' be a primary component of $J = \overline{I}(\Gamma_f) + (Y_{s+1},\ldots,Y_m)$ and assume it is not $\mathfrak{M}_1$-primary and that $\mathfrak{M}_2 \not\subset I'$. This means that there exists a point $a = (a_0 : \ldots : a_n) \in \mathbb{P}^n$ vanishing on all the polynomials of $I' \cap \mathbb{K}[X_0,\ldots,X_n]$. Since this ideal clearly contains $I(X)$ it follows that $(a_0 : \ldots : a_n) \in X$. Let $F'_0,\ldots,F'_n$ represent f locally around $(a_0 : \ldots : a_n)$. As we have remarked in part (i), we can also assume that the minors of

$$\begin{pmatrix} Y_0 & \ldots & Y_m \\ F'_0 & \ldots & F'_m \end{pmatrix}$$

belong to $\overline{I}(\Gamma_f)$, and hence to I'. In particular, since $Y_{s+1},\ldots,Y_m$ are in I' and some other Y_i does not belong to I' (because we are assuming $\mathfrak{M}_2 \not\subset I'$), we conclude that $F'_{s+1},\ldots,F'_m \in \sqrt{I'}$ (because I' is primary). Thus $F'(a_0,\ldots,a_n) = \ldots = F'(a_0,\ldots,a_n) = 0$, which implies that $f(a) \in V(Y_{s+1},\ldots,Y_m)$. But then $(a, f(a)) \in V(J)$, which is a contradiction. This proves (ii) and hence the lemma. $\square$

Theorem 8.15. *Let f be a morphism between projective sets $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ and let $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^m$ be its graph.*

(i) The Hilbert polynomials $P_X \in \mathbb{Q}[T]$ of X in $\mathbb{P}^n$, and $\overline{P}_{\Gamma_f} \in \mathbb{Q}[T_1,T_2]$ of Γ_f in $\mathbb{P}^n \times \mathbb{P}^m$ are related by $P_X(T) = \overline{P}_{\Gamma_f}(T,0)$.

(ii) If f is an isomorphism, then $p_a(X) = p_a(Y)$.

Proof: We already observed that $S(X)_l$ and $\overline{S}(\Gamma_f)_{l,0}$ are canonically isomorphic. Hence it is enough to prove that $\dim \overline{S}(\Gamma_f)_{l,0} = \overline{P}_{\Gamma_f}(l,0)$ for big values of l. We first take a hyperplane (which we assume to be $V(Y_m)$ after changing coordinates) in $\mathbb{P}^m$ not containing any component of the image of f. In other words $V(Y_m)$, as a hypersurface in $\mathbb{P}^n \times \mathbb{P}^m$, does not contain any component of Γ_f. We then have an exact sequence

$$0 \to \overline{S}(\Gamma_f)(0,-1) \xrightarrow{\cdot Y_m} \overline{S}(\Gamma_f) \to \overline{S}/(\overline{I}(\Gamma_f) + (Y_m)) \to 0$$

proving that

$$\dim \overline{S}(\Gamma_f)_{l_1,l_2} - \dim \overline{S}(\Gamma_f)_{l_1,l_2-1} = \dim \bar{(}S/(\overline{I}(\Gamma_f) + (Y_m)))_{l_1,l_2}$$

for any l_1, l_2 (it is important that we do not need to assume l_2 big enough). We just try to repeat now the same procedure by intersecting with another hyperplane of $\mathbb{P}^m$, but we find immediately a problem: $\overline{I}(\Gamma_f) + (Y_m)$ could have embedded components. But thanks to Lemma 8.14(i) we can find another hyperplane, say $V(Y_{m-1})$ such that Y_{m-1} is not in any associated prime of $\overline{I}(\Gamma_f) + (Y_m)$ except maybe in $\mathfrak{M}_1 + \mathfrak{M}_2$. But now, if I' is the $(\mathfrak{M}_1 + \mathfrak{M}_2)$-primary component of $\overline{I}(\Gamma_f) + (Y_m)$ and I_0 is the rest of the primary components, then we have an exact sequence

$$0 \to \overline{S}/I_0(0,-1) \xrightarrow{\cdot Y_{m-1}} \overline{S}/I_0 \to \overline{S}/I_0 + (Y_{m-1})) \to 0$$

And on the other hand the epimorphism $\overline{S}/I_0 \to S/(\overline{I}(\Gamma_f) + (Y_m))$ is an isomorphism in bidegree (l_1,l_2) for l_1 big enough (and again for any l_2), since some power of $\mathfrak{M}_1$ is contained in I'. And hence the same holds for the epimorphism $\overline{S}/(I_0 + (Y_{m-1})) \to S/(\overline{I}(\Gamma_f) + (Y_{m-1}, Y_m))$. As a consequence, we get

$$\dim(\overline{S}/(\overline{I}(\Gamma_f) + (Y_m)))_{l_1,l_2} - \dim \overline{I}(\Gamma_f) + (Y_m)))_{l_1,l_2-1} = \dim \bar{(}S/(\overline{I}(\Gamma_f) + (Y_{m-1}, Y_m)))_{l_1,l_2}$$

for l_1 big enough and any l_2.

We then iterate this process, using always Lemma 8.14(i), until we arrive to some $\overline{I}(\Gamma_f) + (Y_r, \ldots, Y_m)$ defining the empty set. And by Lemma 8.14(ii) all its components either are $\mathfrak{M}_1$-primary or contain $\mathfrak{M}_2$. This means that

$$\dim(\overline{S}/(\overline{I}(\Gamma_f) + (Y_r, \ldots, Y_m)))_{l_1,l_2} = 0$$

for l_1 big enough and any l_2, i.e. the dimension coincides with value of the Hilbert polynomial. But now all the equalities we proved for the dimension, being also valid for the corresponding Hilbert polynomials, show that any time we have $\dim \overline{S}(\Gamma_f)_{l_1,l_2} = \overline{P}_{\Gamma_f}(l_1, l_2)$, it also holds that $\dim \overline{S}(\Gamma_f)_{l_1,l_2-1} = \overline{P}_{\Gamma_f}(l_1, l_2 - 1)$. As a consequence, $\dim \overline{S}(\Gamma_f)_{l_1,0} = \overline{P}_{\Gamma_f}(l_1, 0)$ for big values of l_1, finishing the proof of (i).

Now part (ii) is an easy consequence of (i). Indeed (i) implies that $p_a(X) = p_a(\Gamma_f)$ and $p_a(Y) = p_a(\Gamma_{f^{-1}})$. But clearly Γ_f and $\Gamma_{f^{-1}}$ are isomorphic (just swapping coordinates), and therefore $p_a(\Gamma_f) = p_a(\Gamma_{f^{-1}})$, proving (ii). □

9. Resolutions and dimension

We want to discuss in this chapter the relation between the length of a free resolution of a graded module over $S = \mathbb{K}[X_0, \ldots, X_n]$ and the dimension of its support. Hence the reader who skipped Chapter 4 is invited to either skip also this one or reconsider the possibility of reading that chapter.

To give you an idea of what we want, take a look at the resolutions we got in Chapter 4. In Example 4.5, for the twisted cubic in $\mathbb{P}^3$ we got a resolution of length two, the same of what you should have got in all the resolutions of Exercise 4.6 for sets of points in $\mathbb{P}^2$. Hence the length of the resolution coincides with the codimension. Observe that this is not always the case, since Exercise 4.7 or Exercise 4.8 give examples of curves in $\mathbb{P}^3$ whose coordinate rings have a resolution of length three. On the other hand, if F is a nonconstant homogeneous polynomial of degree d then there is a resolution $0 \to S(-d) \xrightarrow{\cdot F} S \to S/(F) \to 0$, and hence its length is one, which is exactly the codimension of any irreducible component of $V(F)$.

If we want to relate the length of a resolution to the dimension of the support of the module we are studying, it is clear that first of all we will have to reduce ourselves to resolutions that somehow are irredundant. Our first goal is thus to make this idea precise. So assume that we have a free resolution

$$0 \to P_r \to P_{r-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

Taking kernels (as for instance in Theorem 4.3) we have small exact sequences

$$0 \to M_{i+1} \to P_i \xrightarrow{f_i} M_i \to 0$$

for $i = 0, \ldots, r$, in which $M_0 = 0$ and $M_r = P_r$. Each of this small pieces of resolutions corresponds to the choice of a system of homogeneous generators of M_i (namely the image by f_i of a basis of P_i. The resolution will be not optimal if at some point we chose some superfluous generator. Assume for instance that, if $e_1, \ldots, e_s$ is a (homogeneous) basis of P_i, $f_i(e_s)$ can be obtained from $f_i(e_1), \ldots, f_i(e_{s-1})$ as a linear combination $f_i(e_s) = F_1 f_i(e_1) + \ldots + F_{s-1} f_i(e_{s-1})$, with $F_1, \ldots, F_{s-1} \in S$ homogeneous polynomials. Thus we have that $F_1 e_1 + \ldots + F_{s-1} e_{s-1} - e_s$ is an element of $\ker f_i = M_{i+1} = \operatorname{Im} f_{i+1}$ (and it is homogeneous with the graduation defined in P_i). If $e'_1, \ldots, e'_t$ is now a basis of P_{i+1}, we can find thus homogeneous polynomials $G_1, \ldots, G_t \in S$ such that $f_{i+1}(G_1 e'_1 + \ldots + G_t e'_t) = F_1 e_1 + \ldots + F_{s-1} e_{s-1} - e_s$. If A is the matrix defining f_{i+1} we thus have the following identity

$$A \begin{pmatrix} G_1 \\ \vdots \\ G_t \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ F_{s-1} \\ -1 \end{pmatrix}$$

This implies that at least one of the elements of the last row of A (as well as one of the polynomials $G_1, \ldots, G_t$) is a nonzero constant. And in fact it is easy to see that the existence of this constant characterizes the fact that one of the generators $f_i(e_1), \ldots, f_i(e_s)$ can be obtained from the others. This motivates the following definition.

Definition. A *minimal resolution* of a graded S-module M is a free resolution such that none of the matrices defining the maps among the free modules contains nonzero constants, or equivalently if all the elements of those matrices belong to the maximal ideal $\mathfrak{M}$.

Example 9.1. The resolutions in Example 4.5 and Exercise 4.7 (the latter because the numbers force that no element of the matrices of the resolution have degree zero). If you found reasonable resolutions in Exercise 4.6 then you also got minimal resolutions. In particular, for four points in general position you should have obtained a resolution of the type

$$0 \to S(-4) \to S(-2) \oplus S(-2) \to S \to S(X)$$

while if exactly three of the four points lie on a line, then the resolution takes the form

$$0 \to S(-4) \oplus S(-3) \to S(-2) \oplus S(-2) \oplus S(-3) \to S \to S(X)$$

but the part of the first map corresponding to $S(-3) \to S(-3)$ is zero, so that the resolution is indeed minimal.

If we have a non minimal resolution, it is always possible to extract from it a minimal resolution. Keeping the previous notation and assumptions, assume also that G_t is constant (recall that some of the G_i's had to be a nonzero constant). We can thus write $P_{i+1} = P'_{i+1} \oplus P''_{i+1}$, where P'_{i+1} is the free module generated by $e'_1, \ldots, e'_{t-1}$ and P''_i is the free module generated by $G_1 e'_1 + \ldots + G_t e'_t$. Similarly we can write $P_i = P'_i \oplus P''_i$, P'_i being the free module generated by $e_1, \ldots, e_{s-1}$ and P''_i being the module generated by $F_1 e_1 + \ldots + F_{s-1} e_{s-1} - e_s$. We have now that the map $f'_i : P'_i \to M_i$ (restriction of f_i) is still a surjective map, and its kernel is the image of the map $f'_{i+1} : P'_{i+1} \to P'_i$ (restriction of f_{i+1} followed by the natural projection from P_i to its direct summand P'_i. Moreover the kernel of f'_{i+1} is naturally isomorphic to the direct sum of P''_{i+1} and the kernel of f_{i+1}. Therefore still get a free resolution of M if we substitute P_i, P_{i+1}, f_i and f_{i+1} with respectively P'_i, P'_{i+1}, f'_i and f'_{i+1}. Proceeding in such a way as many times as needed, we can remove all the nonzero constants in the matrices of a resolution of any finitely generated module and eventually get a minimal resolution.

We will prove next that there is only one minimal resolution up to isomorphism. To that purpose, let us study first isomorphisms among free modules.

Lemma 9.2. *Let $P=\bigoplus_{i=1}^{r} S(-a_i)$ and $Q=\bigoplus_{j=1}^{s} S(-b_j)$ be two free modules. Then the following are equivalent:*

(i) P and Q are isomorphic.

(ii) $r=s$ and there is a map $P\to Q$ defined by a matrix whose determinant is a nonzero constant.

(iii) $r=s$ and, after probably permuting the degrees, $a_i=b_i$ for $i=1,\dots,r$.

Moreover, a morphism between two isomorphic free modules is an isomorphism if and only if it is injective. More precisely, in case $\varphi: P\to Q$ is not an isomorphism there is an r-uple with some nonzero constant coordinate that is in $\ker\varphi$.

Proof: We prove the equivalences in a cyclic way.

$(i)\Rightarrow(ii)$ Let $\varphi: P\to Q$ be an isomorphism, and let A and B be respectively be the matrices associated to A and B. Since AB is the $s\times s$ identity matrix, it follows that A has rank at least s when substituting the indeterminates $X_0,\dots,X_n$ by the coordinates of any vector in $\mathbb{K}^{n+1}$. Therefore $s\le r$. Similarly, BA is the $r\times r$ identity matrix, so that $r\le s$ and hence $r=s$. Now $\det(A)\det(B)=1$, which implies that $\det(A)$ is a nonzero constant.

$(ii)\Rightarrow(iii)$ Let A be the matrix of a morphism $P\to Q$ such that $\det(A)$ is a nonzero constant. It thus follows that some of the summands defining the determinant is also a nonzero constant. By permuting the values of $b_1,\dots,b_r$ we can assume that the product of the elements of the diagonal of A is a nonzero constant. This means that all the entries of the diagonal have degree zero. But the degree of the (i,i)-entry if A is a_i-b_i, so that $a_i=b_i$ for $i=1,\dots,r$.

$(iii)\Rightarrow(i)$ This is trivial.

For the last statement, it is enough to prove that a morphism φ from a free module P to itself that is not an isomorphism has an element in the kernel as in the statement. So it is convenient to re-write P as $P=\bigoplus_{i=1}^{s} S(-a_i)^{r_i}$, with $a_1<\ldots<a_s$ and let A be the matrix representing φ. We can write it in the way

$$A=\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1s}\\ 0 & A_{22} & \dots & A_{2s}\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & A_{ss}\end{pmatrix}$$

where each A_{jk} is an $r_k\times r_j$ matrix of homogeneous forms of degree a_k-a_j. Since $\det(A)=\det(A_{11})\cdots\det(A_{ss})\in\mathbb{K}$, the first part of the lemma proves that some $\det(A_{ii})$ is zero. We can thus take the maximum $t\in\{1,\dots,s\}$ such that $\det(A_{tt})=0$. We consider now the free submodules $P_t := \bigoplus_{i=t}^{s} S(-a_i)^{r_i} \subset P$ and $P_{t+1} := \bigoplus_{i=t+1}^{s} S(-a_i)^{r_i} \subset P_t$. We have that P_{t+1} and P_t are invariant by φ. The restriction of φ to P_{t+1} is given by

$$\begin{pmatrix} A_{t+1,t+1} & A_{t+1,t+2} & \dots & A_{t+1,s} \\ 0 & A_{t+2,t+2} & \dots & A_{t+2,s} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A_{ss} \end{pmatrix}$$

so that it is an isomorphism (again by the first part of the lemma). Similarly, the restriction of φ to $P_t = S(-a_t)^{r_t} \oplus P_{t+1}$ is not an isomorphism, and our goal is to find an element of its kernel having at least one coordinate in the summand $S(-a_t)^{r_t}$ that is a nonzero constant. But this is easy, since A_{tt} is now an $r_t \times r_t$ matrix (of constant entries) whose determinant is zero, and hence there exists a nonzero vector $u \in \mathbb{K}^{r_t}$ such that $A_{tt}u = 0$. We then consider $(u, 0) \in P_t = S(-a_t)^{r_t} \oplus P_{t+1}$, and have that $\varphi(u, 0) = (0, v')$, with $v' \in P_{t+1}$. Since φ is an isomorphism when restricted to P_{t+1}, there exists $v \in P_{t+1}$ such that $\varphi(v) = v'$. The wanted element in the kernel of φ is thus $(u, -v)$ (you should add some zeros if you want to regard it as an uple in P. This completes the proof of the lemma. $\square$

Theorem 9.3. *Let M, M' be two finitely generated graded S-modules and let*

$$0 \to P_r \xrightarrow{f_r} P_{r-1} \to \dots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$
$$0 \to P'_r \xrightarrow{f'_0} P'_{r-1} \to \dots \to P'_1 \xrightarrow{f'_1} P'_0 \xrightarrow{f'_0} M' \to 0$$

be two free resolutions (we will allow some of the modules in the resolution to be zero, so that both resolutions have the same length r).

(i) If there is a graded homomorphism $\varphi : M \to M'$, then there are maps $\varphi_i : P_i \to P'_i$ such that the diagram

$$\begin{array}{ccccccccccc} 0 \to & P_r & \to & P_{r-1} & \to \dots \to & P_1 & \to & P_0 & \to & M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & P'_r & \to & P'_{r-1} & \to \dots \to & P'_1 & \to & P'_0 & \to & M' & \to 0 \end{array}$$

is commutative.

(ii) (Mapping cylinder construction) *If φ is injective, the module $M'/\varphi(M)$ has a resolution*

$$0 \to P_r \to P_{r-1} \oplus P'_r \to \dots \to P_0 \oplus P'_1 \to P'_0 \to M'/\varphi(M) \to 0$$

(iii) If $M = M'$ and φ is the identity, and both resolutions of M are minimal, then each φ_i is an isomorphism (and hence all the minimal resolutions of M have the same length, which is at most $n + 1$).

Proof: Part (i) is obtained from a simple diagram chasing. Fix a basis $\{e_1, \dots, e_s\}$ of P_0. Since $f'_0 : P'_0 \to M'$ is surjective, we can choose $u'_1, \dots, u'_s \in P'_0$ such that $f'_0(u'_i) =$

$\varphi(f_0(e_i))$, for $i = 1, \ldots, s$. We can thus define $\varphi_0 : P_0 \to P'_0$ by $\varphi_0(e_i) = u'_i$. It is immediate to see that $\varphi_0(\ker f_0) \subset \ker f'_0$, so that we can apply the above construction to the epimorphisms $P_1 \to \ker f_0$ and $P'_1 \to \ker f'_0$, getting now $\varphi_1 : P_1 \to P'_1$. Iterating the process we arrive to the wanted commutative diagram.

Let us prove now (ii). It is clear that the composition $\psi : P'_0 \to M' \to M'/\varphi(M)$ is an epimorphism. Assume $v'_0 \in P'_0$ is in the kernel of ψ. This means that $f'_0(v'_0) = \varphi(m)$, for some $m \in M$. Since f_0 is surjective, there exists $v_0 \in P_0$ such that $m = f_0(v_0)$. Therefore $f'_0(v'_0) = \varphi(f_0(v_0)) = f'_0(\varphi_0(v_0))$ and thus $v'_0 - \varphi_0(v_0) \in \ker f'_0 = \operatorname{Im} f'_1$. We find thus $v'_1 \in P'_1$ such that $v'_0 - \varphi_0(v_0) = f'_1(v'_1)$. Defining $\psi_0 : P_0 \oplus P'_1 \to P'_0$ by $\psi_0(v_0, v'_1) = \varphi_0(v_0) + f'_1(v'_1)$ we then see that $\ker \psi = \operatorname{Im} \psi_0$. If $\psi(v_0, v'_1) = 0$, then $0 = f'_0(\varphi_0(v_0) + f'_1(v'_1)) = \varphi(f_0(v_0))$. Since φ is injective, we get that $v_0 \in \ker f_0 = \operatorname{Im} f_1$, and therefore there exists $v_1 \in P_1$ such that $v_0 = f_1(v_1)$. We thus have $0 = \varphi_0(f_1(v_1)) + f'_1(v'_1) = f'_1(\varphi(v_1) + v'_1)$. We thus find now $v'_2 \in P'_2$ such that $\varphi(v_1) + v'_1 = f'_2(v'_2)$. In other words, we found the equality $(v_0, v'_1) = (f_1(v_1), -\varphi(v_1) + f'_2(v'_2))$.

We define in general (for $i = 0, \ldots, r$) the maps $\psi_i : P_i \oplus P'_{i+1} \to P_{i-1} \oplus P'_i$ by $\psi_i(v_i, v'_{i+1}) = (f_i(v_i), (-1)^i \varphi_i(v_i) + f'_{i+1}(v'_{i+1}))$ (we write $P'_j = P_j = 0$ if $j \notin \{0, \ldots, r\}$). We clearly have that $\psi_i \circ \psi_{i+1} = 0$. On the other hand, if $\psi(v_i, v'_{i+1}) = 0$, then $f_i(v_i) = 0$ and $f'_{i+1}(v'_{i+1}) = (-1)^{i+1}\varphi_i(v_i)$. From the first equality we get that there exists $v_{i+1} \in P_{i+1}$ such that $v_i = f_{i+1}(v_{i+1})$, and substituting this in the second equality we have $f'_{i+1}(v'_{i+1}) = (-1)^{i+1}\varphi_i(f_{i+1}(v_{i+1})) = (-1)^{i+1} f'_{i+1}\varphi_{i+1}(v_{i+1})$. Therefore $v'_{i+1} - (-1)^{i+1}\varphi_{i+1}(v_{i+1})$ is in the kernel of f'_{i+1}, so that it can be written as $f'_{i+2}(v'_{i+2})$ for some $v'_{i+2} \in P'_{i+2}$. Hence $(v_i, v'_{i+1}) = \psi_{i+1}(v_{i+1}, v'_{i+2})$. This shows that we have an exact sequence as in the statement of part (ii).

For part (iii), observe that from part (i) we have maps $\varphi_i : P_i \to P'_i$ such that $\varphi_{i-1} \circ f_i = f'_i \circ \varphi_i$ (we can write $\varphi_{-1} := id_M$) and maps $\psi_i : P'_i \to P_i$ such that $\psi_{i-1} \circ f'_i = f_i \circ \psi_i$.

Let us prove that each $\psi_i \circ \varphi_i$ is injective by induction on i, the case $i = -1$ being trivial. Assume $\psi_i \circ \varphi_i(v_i) = 0$ for some $v_i \in P_i$. Then $\psi_{i-1} \circ \varphi_{i-1} f_i(v_i) = 0$, and thus by induction hypothesis $f_i(v_i) = 0$. Therefore there exists $v_{i+1} \in P_{i+1}$ such that $v_i = f_{i+1}(v_{i+1})$. Since the resolution for M was minimal, this implies that v_i is an uple of homogeneous polynomials, none of them a nonzero constant, and this holds for any v_i in $\ker \psi_i \circ \varphi_i$. But the last statement of Lemma 9.2 implies that, if $\psi_i \circ \varphi_i$ is not injective, then we can find in $\ker(\psi_i \circ \varphi_i)$ an uple with some nonzero constant as one of its coordinates. This proves by contradiction that $\psi_i \circ \varphi_i$ is injective.

In a similar way, every $\varphi_i \circ \psi_i$ is also injective. But now Lemma 9.2 implies that both $\psi_i \circ \varphi_i$ and $\varphi_i \circ \psi_i$ are isomorphisms. This implies respectively that φ_i is injective and surjective, and hence an isomorphism, as wanted.

The fact that the length of a minimal resolution is at most $n+1$ is an immediate consequence of the Hilbert's syzygy Theorem (Theorem 4.3) and the fact that from any free resolution we can extract a minimal resolution. $\square$

Definition. If $0 \to P_r \xrightarrow{f_r} P_{r-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ is a minimal resolution of M, the module $M_i := \ker f_{i-1}$, $i = 1, \ldots, r+1$ is called the *i-th syzygy module* of M.

Remark 9.4. Unfortunately, under the hypothesis of Theorem 9.3 it is not true that the injectivity of φ implies the injectivity of the maps φ_i (which would have considerably simplify the proof of (iii)). Although there are numerous examples of this fact (take for instance M' to be S and let M be any non-principal ideal), let us show a more clarifying example (from the point of view of the result we are looking for). Let $X \subset \mathbb{P}^3$ be the disjoint union of two lines L_1 and L_2. We obviously have an inclusion $S(X) \hookrightarrow S(L_1) \oplus S(L_2)$, whose quotient is $S/\mathfrak{M}$ (see Lemma 3.1). Exercise 4.8 provides us resolutions $0 \to S(-4) \to S(-3)^4 \to S(-2)^4 \to S \to S(X) \to 0$, and $0 \to S(-2)^2 \to S(-1)^4 \to S^2 \to S(L_1) \oplus S(L_2) \to 0$. We thus have a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \to & S(-4) & \to & S(-3)^4 & \to & S(-2)^4 & \to & S & \to & S(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & S(-2)^2 & \to & S(-1)^4 & \to & S^2 & \to & S(L_1) \oplus S(L_2) & \to & 0
\end{array}$$

in which obviously the first vertical arrows cannot be injective. It is not difficult, however, to see that the map cylinder construction provides (after removing one redundant summand S) the following resolution

$$0 \to S(-4) \to S(-3)^4 \to S(-2)^6 \to S(-1)^4 \to S \to S/\mathfrak{M} \to 0$$

This last resolution is a particular case of what we are going to obtain now.

Proposition 9.5. *Let $F_1, \ldots, F_r \in S = \mathbb{K}[X_0, \ldots, X_n]$ be homogeneous polynomials of respective degrees $d_1, \ldots, d_r$. If for each $i = 1, \ldots, r$ it holds that F_i is not in any associated prime of $(F_1, \ldots, F_{i-1})$, then there is a minimal free resolution of $S/(F_1, \ldots, F_r)$ of the type*

$$0 \to P_r \xrightarrow{f_r} P_{r-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} S/(F_1, \ldots, F_r) \to 0$$

where $P_i = \bigoplus_{1 \le j_1 < \ldots < j_i \le r} S(-d_{j_1} - \ldots - d_{j_i})$ (we write $P_0 = S$) and, if $\{e_{j_1 \ldots j_i}\}$ is a basis of P_i then $f_i(e_{j_1 \ldots j_i}) = \sum_{k=1}^{i} (-1)^k F_{j_k} e_{j_1 \ldots \hat{j_k} \ldots j_i}$ (a hat over an index means that it has been removed).

Proof: We prove it by induction on r. If $r = 1$, the statement is just saying that we have the known exact sequence $0 \to S(-d_1) \xrightarrow{\cdot F_1} S \to S/(F_1) \to 0$.

If we have now the result to be true for $r-1$, we thus have a resolution

$$0 \to P'_{r-1} \xrightarrow{f'_{r-1}} P'_{r-2} \to \ldots \to P'_0 \xrightarrow{f'_0} S/(F_1, \ldots, F_{r-1}) \to 0$$

with $P'_i = \bigoplus_{1 \le j_1 < \ldots < j_i \le r-1} S(-d_{j_1} - \ldots - d_{j_i})$. Shifting degrees, we also get

$$0 \to P'_{r-1}(-d_r) \xrightarrow{f'_{r-1}} P'_{r-2}(-d_r) \to \ldots \to P'_0(-d_r) \xrightarrow{f'_0} \big(S/(F_1, \ldots, F_{r-1})\big)(-d_r) \to 0$$

Since F_r in not in any associated prime of $(F_1, \ldots, F_{r-1})$ we thus get from Lemma 3.12 that there is an exact sequence

$$0 \to \big(S/(F_1, \ldots, F_{r-1})\big)(-d_r) \xrightarrow{\cdot F_r} S/(F_1, \ldots, F_{r-1}) \to S/(F_1, \ldots, F_r) \to 0$$

Obviously this map induces a map between the two resolutions consisting of the multiplication by F_r. The mapping cylinder construction (Theorem 9.3(ii)) provides now a free resolution

$$0 \to P_r \xrightarrow{f_r} P_{r-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} S/(F_1, \ldots, F_r) \to 0$$

where $P_i = P'_i \oplus P'_{i-1}(-d_r) = \bigoplus_{1 \le j_1 < \ldots < j_i \le r} S(-d_{j_1} - \ldots - d_{j_i})$ (if in the last direct sum you consider the summands in which $j_i \neq r$ you get P'_i, while those for which $j_i = r$ provide $P'_{i-1}(-d_r)$). Finally, let us see that the map $P_i \to P_{i-1}$ is the one of the statement. So let $e_{j_1 \ldots j_i}$ an element of the basis of P_i. If $j_i \neq r$, then $e_{j_1 \ldots j_i}$ can be regarded as an element of P'_i, and hence the proof of Theorem 9.3(ii) tells us that is image is $f'_i(e_{j_1 \ldots j_i})$, which by induction is indeed $\sum_{k=1}^{i} (-1)^k F_{j_k} e_{j_1 \ldots \hat{j_k} \ldots j_i}$. If instead $j_i = r$, then $e_{j_1 \ldots j_i}$ can be identified with $e_{j_1 \ldots j_{i-1}} \in P'_{i-1}(-d_r)$. But again the proof of Theorem 9.3(ii) tells us that its image by f_i must be the sum of $(-1)^i$ times the image in P'_{i-1} by the multiplication by F_r plus its image by f'_i in $P'_{i-2}(-d_r)$. Again by induction hypothesis and the natural identification of P_{i-1} as a sum of two pieces we get $f(e_{j_1 \ldots j_i}) = (-1)^i F_r e_{j_1 \ldots j_{i-1} r} + \sum_{k=1}^{i-1} (-1)^k F_{j_k} e_{j_1 \ldots \hat{j_k} \ldots j_{i-1} r}$, which is the wanted formula. The fact that the resolution is minimal comes from the fact that the entries of the matrices defining the maps f_i are either zero or (up to a sign) any of the polynomials $F_1, \ldots, F_r$. □

Definition. A set of polynomials $F_1, \ldots, F_r$ as in the hypothesis of the above Proposition is called a *regular sequence*. The resolution that we found for $S/(F_1, \ldots, F_r)$ is called the *Koszul exact sequence associated to the regular sequence.*

Example 9.6. If we take $F_1, \ldots F_r$ to be any set of linearly independent linear forms, then they form a regular sequence. Therefore we have a minimal resolution

$$0 \to S(-r) \to S(-r-1)^r \to \ldots \to S(-1)^r \to S \to S/(F_1, \ldots, F_r) \to 0$$

(the number of copies of $S(-i)$ is $\binom{r}{i}$). If $r \leq n$, this gives a minimal resolution of the graded ring of the linear space $V(F_1, \ldots, F_r)$, and its length is thus the codimension of this linear space. If $r = n+1$, then the ideal $(F_1, \ldots, F_{n+1})$ is the maximal ideal $\mathfrak{M}$, so that $S/(F_1, \ldots, F_r) \cong \mathbb{K}$. The projective set is this time empty, and although $\mathfrak{M}$ is not its homogeneous ideal, the length of the resolution we found is the maximum allowed by the Hilbert's syzygy Theorem, namely $n+1$ (which is in fact the codimension of the empty set, considered as a set of dimension -1).

Exercise 9.7. Find explicitly and prove by hand that there is an exact sequence $0 \to S(-3) \to S(-2)^3 \to S(-1)^3 \to S \to S/(X_0, X_1, X_2) \to 0$ in any $S = \mathbb{K}[X_0, \ldots, X_n]$ with $n \geq 2$.

Proposition 9.8. *A graded S-module M ($S = \mathbb{K}[X_0, \ldots, X_n]$) has a minimal resolution of length $n+1$ if and only if M has an $\mathfrak{M}$-primary component.*

Proof: Let $0 \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ be a minimal resolution of M. If X_n is not a zerodivisor of M, and write $\overline{P}_i = P_i/X_nP_i$, we have that $0 \to \overline{P}_{n+1} \to \overline{P}_n \to \ldots \to \overline{P}_1 \to \overline{P}_0 \to M/X_nM \to 0$ is a free resolution of M/X_nM as a module over $S/(X_n) \cong \mathbb{K}[X_0, \ldots, X_{n-1}]$ (see Lemma 4.2 and its application in Theorem 4.3). On the other hand, it is clear that the resolution is minimal, since reducing a matrix modulo X_n can never produce nonzero constants. But this is a contradiction, since the length of a minimal resolution in $\mathbb{K}[X_0, \ldots, X_{n-1}]$ can never exceed n.

This shows that X_n, and by the same reason any linear form, is a zerodivisor of M. This implies that $\mathfrak{M}$ is contained in the set of zerodivisors of M. By Proposition 4.12, the set of zerodivisors of M is the union of the associated primes of a primary decomposition of M. By Exercise 0.1(vi) we thus have that $\mathfrak{M}$ is contained in some associated prime of M, and by its maximality it is therefore one of them. Thus there is an $\mathfrak{M}$-primary component of M.

Reciprocally, if M has an $\mathfrak{M}$-primary component, then by Proposition 4.12 we know that there exists $m \in M$ such that $Ann(m) = \mathfrak{M}$. This means, if m has degree d, that we have an injection $\varphi : S/\mathfrak{M}(-d) \to M$ by assigning to the class of $F \in S$ the product Fm. Taking the minimal resolution

$$0 \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to S/\mathfrak{M} \to 0$$

found in Example 9.6 (conveniently shifted) and a minimal resolution

$$0 \to P'_{n+1} \to P'_n \to \ldots \to P'_1 \to P'_0 \to M \to 0$$

of M (in which we add as many $P'_i = 0$ as needed), Theorem 9.3(ii) provides a free resolution

$$0 \to P_{n+1} \to P_n \oplus P'_{n+1} \to \ldots \to P_0 \oplus P'_1 \to P'_0 \to M/(x) \to 0$$

Since it has length $n+2$, it follows that P_{n+1} is redundant, so that the matrix defining $P_{n+1} \to P_n \oplus P'_{n+1}$ has some nonzero constants. Since the resolution of $S/\mathfrak{M}$ was minimal, we conclude that the matrix of $P_{n+1} \to P'_{n+1}$ has nonzero constants. In particular, P'_{n+1} is not zero, and hence the resolution of M has length $n+1$. $\square$

Example 9.9. Observe that Proposition 9.8 implies that the curve of Exercise 4.7 does not have good hyperplane sections. Indeed any linear form H not vanishing at X will produce a minimal resolution of $\mathbb{K}[X_0, X_1, X_2, X_3]/(I(X) + (H))$ of length three, and hence $I(X) + (H)$ will always have embedded components. Hence the behavior of the hyperplane chosen in Example 1.22 was not special for the curve, but the general one.

Theorem 9.10. *If a graded module M has a minimal resolution $0 \to P_r \to P_{r-1} \to \ldots \to P_1 \to P_0 \to M \to 0$, then for any prime I in the support of M it holds $\dim V(I) \geq n - r$.*

Proof: By the Hilbert's syzygy Theorem, the smallest possible value is $n-r = -1$. But this case is trivial. The case $n = r$ comes from Proposition 9.8, since a prime with $\dim V(I) < 0$ is necessarily $\mathfrak{M}$. So assume $r \leq n-1$, and suppose for contradiction that we have some associated prime I of M with $\dim V(I) < n - r$. Thus from Proposition 9.8 we can take $F_1 \in S$ that is not a zerodivisor of M. In fact, it is enough to take F_1 not belonging to any associated prime to M (in particular, $F_1 \notin I$). By Theorem 9.3(ii) we get a resolution of M/F_1M of length $r+1$. So we could iterate this $n+1-r$ times until we arrive to a minimal resolution of length $n+1$. But we will have to proceed carefully.

Specifically, given $F_1, \ldots, F_i$ with $i < n - r$, we will take ${}_{i+1}$ not only with the condition that F_{i+1} is not a zerodivisor of $M/(F_1, \ldots, F_i)M$ –which guarantees by Theorem 9.3(ii)that $M/(F_1, \ldots, F_{i+1})M$ will have a minimal resolution of length $r+i+1$– but also that F_{i+1} does not belong to any relevant associated prime of $I + (F_1, \ldots, F_i)$. This is just imposing F_{i+1} not to belong to a finite union of homogeneous prime ideals different from $\mathfrak{M}$, so it is a non-empty condition.

We eventually get $F_1, \ldots, F_{n-r}$ such that $M/(F_1, \ldots, F_{n-r})M$ has a minimal resolution of length n, and hence from Proposition 9.8 it should have no irrelevant component. But let us see that the choice of $F_1, \ldots, F_t$ forces the existence of such component. Indeed, form our choice we have that $\dim V(I) \cap V(F_1, \ldots, F_{i+1}$ is either -1 or $\dim V(I) \cap V(F_1, \ldots, F_i - 1$. Since $\dim V(I) < n-r$, we get that $V(I) \cap V(F_1, \ldots, F_{n-r}) = \emptyset$. By the Weak Nullstellensatz (Theorem 1.24), it follows that a power of $\mathfrak{M}$ is contained in $I + (F_1, \ldots, F_{n-r})$.

On the other hand, the fact that I was an associated prime of M means that there exists $m \in M$ such that $I = Ann(m)$. We would like to keep m different from zero when quotienting at each step. This is not completely possible but almost. In fact, if $m \in F_1M$, then we can write $m = F_1 m'$ for some other $m' \in M$. I claim that $I = Ann(m')$. Indeed, one inclusion is obvious, and for the other, if $G \in I$, then $0 = Gm = F_1 G m'$; but since F_1 was not a zerodivisor of M it follows that $G \in Ann(m')$. Since $\deg m' < \deg m$, after a finite number of steps we will be able to write I as the annihilator of an element whose class modulo $F_1 M$ is not zero. Iterating this, we can assume that $I = Ann(m)$, such that m is not zero modulo $(F_1, \ldots, F_{n-r})M$.

Putting together what we got in the last two remarks, we see that there is a nonzero element $\overline{m} \in M/(F_1, \ldots, F_{n-r})M$ that is vanished by powers of $X_0, \ldots, X_n$. Changing $\overline{m}$ by $X_0^t \overline{m}$, where t is the maximum power of X_0 that does not kill $\overline{m}$, we can assume that $X_0 \overline{m} = 0$. Similarly, we can assume $X_i \overline{m} = 0$ for $i = 0, \ldots, n$ and $\overline{m} \neq 0$. This means that $\mathfrak{M} \subset Ann(\overline{m}) \subsetneq S$. Therefore $Ann(\overline{m}) = \mathfrak{M}$, and Proposition 4.12 implies now that $M/(F_1, \ldots, F_{n-r})M$ has an irrelevant primary component. This yields the wanted contradiction and finishes the proof of the theorem. □

Proposition 9.11. *Let $F_1, \ldots, F_r \in S$ be non constant homogeneous polynomials. Then the following are equivalent:*

(i) $F_1, \ldots, F_r$ *form a regular sequence.*

(ii) The Koszul sequence constructed in Proposition 9.5 is exact.

(iii) All the primary components of the ideal $(F_1, \ldots, F_r)$ *have dimension* $n - r$.

(iv) $\dim V(F_1, \ldots, F_r) = n - r$.

Proof: We will use induction on r and will prove the equivalences in a cyclic way.

$(i) \Rightarrow (ii)$ This is proved in Proposition 9.5.

$(ii) \Rightarrow (iii)$ It can be seen from the Koszul exact sequence that the Hilbert polynomial of $S/(F_1, \ldots, F_r)$ has degree $n - r$ (in fact it is even simpler to use induction on r and/or use the mapping cylinder construction). This shows that all the primary components of $(F_1, \ldots, F_r)$ have dimension at most $n - r$. On the other hand, Theorem 9.10 shows that no primary component can have smaller dimension.

$(iii) \Rightarrow (iv)$ It is trivial

$(iv) \Rightarrow (i)$ Since from Proposition 5.7(viii) we have $\dim V(F_1, \ldots, F_{r-1}) \geq n - r + 1$, it necessarily follows that $\dim V(F_1, \ldots, F_{r-1}) = n - r + 1$ (now from part (ii) of that proposition). We can thus apply induction hypothesis, and therefore $F_1, \ldots, F_{r-1}$ form a regular sequence and all the primary components of $V(F_1, \ldots, F_{r-1})$ have dimension $n - r + 1$. Since $\dim V(F_1, \ldots, F_r) = n - r$, it thus follows that F_r dos not vanish on any component defined by those primary components. In other words, $F_1, \ldots, F_r$ form a regular sequence. □

Definition. An ideal generated by a set of polynomial $F_1, \dots, F_r$ like in Proposition 9.11 is called a *complete intersection ideal.* Observe that the order of the polynomials is indifferent, and that part (iii) implies in particular that the ideal has no embedded components. A *complete intersection set* is a projective set whose homogeneous ideal is a complete intersection.

Example 9.12. Any set of points in a line form a complete intersection. Also four points in $\mathbb{P}^2$ in general position are the complete intersection of two conics. A set of three points in $\mathbb{P}^2$ in general position is not a complete intersection, even if it can be defined by two polynomials. Indeed $V(X_1X_2, X_0X_1 + X_0X_2) = \{(1:0:0), (0:1:0), (0:0:1)\}$, but its homogeneous ideal is (X_0X_1, X_0X_2, X_1X_2), which cannot be generated by two polynomials. This is what is called a *set-theoretical complete intersection.*

10. Ruled varieties

In this section we will use ruled varieties (i.e. varieties defined as union of linear spaces) as a sample of how to use the results of Chapter 7. At the end we will give many results only in a very intuitive way (some of them will be mentioned in the construction of the Chow variety). The reader who is not quite convinced of reading this last part has my permission to skip it, since it has been written essentially to my own pleasure.

The main construction of this chapter (which is a more rigorous definition of ruled variety) is given in (the proof of) the following lemma.

Lemma 10.1. *Let $Z \subset \mathbb{G}(k,n)$ be a variety of dimension r. Then the union in $\mathbb{P}^n$ of all the k-planes parametrized by Z is an irreducible projective set $X \subset \mathbb{P}^n$ of dimension at most $r+k$. Moreover, the dimension of X is exactly $r+k$ if and only if through a general point of X there pass only a finite number of k-planes of Z.*

Proof: We just need to consider the incidence set $\Sigma_Z \subset Z \times \mathbb{P}^n$ consisting of pairs (Λ, x) such that $x \in \Lambda$. This is clearly a projective set inside $\mathbb{G}(k,n) \times \mathbb{P}^n$, and considering the projection onto the first factor Theorem 8.4(iii) implies that Σ_Z is irreducible of dimension $r+k$. But now Proposition 7.18 applied to the second projection of Σ_Z to $\mathbb{P}^n$ (whose image is by definition X) implies that X is irreducible of dimension at most $r+k$. But using again Theorem 8.4 (now for this second projection) we immediately obtain the last part of the statement. □

Example 10.2. Let $X, Y \subset \mathbb{P}^n$ be two projective varieties. Consider the map $f: X \times Y \to \mathbb{G}(1,n)$ that associates to each pair of points the line passing through them. This is a regular map (if X and Y meet there will be some points in which f is not defined, but we can always use Lemma 7.8(iii) to replace f with a well-defined map). Then the image of f is a projective variety $J_{X,Y} \subset \mathbb{G}(1,n)$. The fiber of f has positive dimension only for very particular choices of X and Y. For instance, if we can find points $x \in X \setminus Y$ and $y \in Y \setminus X$ (i.e. if $X \not\subset Y$ and $Y \not\subset X$), then the fiber of the line $L = <x, y>$ is finite, since $L \cap X$ and $L \cap Y$ are necessarily finite. Hence in general $\dim J_{X,Y} = \dim X + \dim Y$. Thus the union $J(X,Y)$ of the lines of $J_{X,Y}$ (called the *join variety* of X and Y) is a variety of dimension at most $\dim X + \dim Y + 1$. The following proposition shows that we can say a lot if X and Y are contained in skew linear spaces.

Proposition 10.3. *Let $X, Y \subset \mathbb{P}^n$ be projective varieties and assume there exists $m \in \mathbb{N}$ such that $X \subset V(X_{m+1}, \dots, X_n)$ and $Y \subset V(X_0, \dots, X_m)$.*

(i) A point $(a_0 : \dots : a_n)$ belongs to $J(X,Y)$ if and only if $(a_0 : \dots : a_m : 0 \dots : 0) \in X$ and $(0 : \dots : 0 : a_{m+1} : \dots : a_n) \in Y$.

(ii) The Hilbert function of $J(X,Y)$ *is given by* $h_{J(X,Y)}(l) = \sum_{i=0}^{l} h_X(i)h_Y(l-i)$.

(iii) $\dim J(X,Y) = \dim X + \dim Y + 1$ *and* $\deg J(X,Y) = \deg X \cdot \deg Y$.

Proof: Part (i) is an immediate consequence of the fact that the point $(a_0 : \ldots : a_n)$ is only in one line meeting both $V(X_{m+1},\ldots,X_n)$ and $V(X_0,\ldots,X_m)$, namely the line spanned by $(a_0 : \ldots : a_m : 0 \ldots : 0)$ and $(0 : \ldots : 0 : a_{m+1} : \ldots : a_n)$.

The proof of part (ii) is quite similar to the one of Proposition 6.14. A first difference is that we write now any homogeneous polynomial $F \in \mathbb{K}[X_0,\ldots,X_n]$ as a sum $\sum P_tQ_t$ with $P_t \in \mathbb{K}[X_0,\ldots,X_m]$ and $Q_t \in \mathbb{K}[X_{m+1},\ldots,X_n]$ still homogeneous. We choose thus for each $i = 0,\ldots,l$ a basis $\{A_{ij}\}_j$ of $\mathbb{K}[X_0,\ldots,X_m]_i$ modulo $I(X) \cap \mathbb{K}[X_0,\ldots,X_m]$ (hence its cardinality is $h_X(i)$) and a basis $\{B_{it}\}_t$ of $\mathbb{K}[X_{m+1},\ldots,X_n]_i$ modulo $I(Y) \cap \mathbb{K}[X_{m+1},\ldots,X_n]$ (of cardinality $h_Y(i)$). Then clearly the classes of the products $A_{ij}B_{l-i,t}$ (when i,j,t vary) define a system of generators of $S(J(X,Y))_l$ (observe that part (i) implies that the elements of $I(X)\cap\mathbb{K}[X_0,\ldots,X_m]$ and $I(Y)\cap\mathbb{K}[X_{m+1},\ldots,X_n]$ are in $I(J(X,Y))$).

We just need to prove that the above generators actually form a basis. Assume thus that there is a linear combination $\sum_{ijt}\lambda_{ijt}A_{ij}B_{l-i,t}$ whose class modulo $I(J(X,Y))$ is zero. Then, for any $(a_0 : \ldots : a_m : 0 \ldots : 0) \in X$ we have that $\sum_{ijt}\lambda_{ijt}A_{ij}(a_0,\ldots,a_m)B_{l-i,t}$ belongs to $I(Y)\cap\mathbb{K}[X_{m+1},\ldots,X_n]$. Since this is a homogeneous ideal in $\mathbb{K}[X_{m+1},\ldots,X_n]$, we have that for each $i = 0,\ldots,l$ the homogeneous polynomial $\sum_{jt}\lambda_{ijt}A_{ij}(a_0,\ldots,a_m)B_{l-i,t}$ is in $I(Y) \cap \mathbb{K}[X_{m+1},\ldots,X_n]$. But since the set $\{B_{l-i,t}\}_t$ is linearly independent modulo $I(Y) \cap \mathbb{K}[X_{m+1},\ldots,X_n]$, it follows that for each i,t we have $\sum_j \lambda_{ijt}A_{ij}(a_0,\ldots,a_m) = 0$. This holds for any $(a_0 : \ldots : a_m : 0 \ldots : 0) \in X$, so that $\sum_j \lambda_{ijt}A_{ij}$ is in $I(X) \cap \mathbb{K}[X_0,\ldots,X_m]$. Using now that the set $\{A_{ij}\}_j$ is linearly independent modulo $I(X) \cap \mathbb{K}[X_0,\ldots,X_m]$ we eventually get that $\lambda_{ijt} = 0$ for any i,j,t, thus finishing the proof of (ii).

In order to prove (iii) we cannot apply immediately (ii), since $h_X(i)$ and $h_Y(i)$ coincides respectively with $P_X(i)$ and $P_Y(i)$ only for large values of i, say for $i \geq i_0$ (and small values of i are always involved in the formula for $h_{J(X,Y)}(l)$ independently of the value of l). But we know that, for $l >> 0$ $h_{J(X,Y)}(l)$ is given by a polynomial (of degree $\dim X + \dim Y + 1$, since we have remarked in Example 10.2 that this is the dimension of the join of X and Y). Hence substituting for $i < i_0$ $h_X(i)$ and $h_Y(i)$ with respectively $P_X(i)$ and $P_Y(i)$ in the formula for $h_{J(X,Y)}(l)$ in (ii), the difference with the Hilbert polynomial $P_{J(X,Y)}(l)$ is a polynomial of degree strictly less than the degree of $P_{J(X,Y)}(l)$. It is enough then to find the leading coefficient of $\sum_{i=0}^{l} P_X(i)P_Y(l-i)$.

Writing $r := \dim X$, $s := \dim Y$, $d := \deg X$ and $e := \deg Y$, we have that $P_X(i) = \frac{d}{r!}i^r +$ terms of lower degree in i, and $P_Y(i) = \frac{e}{s!}i^s +$ terms of lower degree in i. Therefore the term of maximum degree with respect to l in $\sum_{i=0}^{l} P_X(i)P_Y(l-i)$ will be the one of $\sum_{i=0}^{l} \frac{d}{r!}i^r \frac{e}{s!}(l-i)^s$. But this term of maximum degree coincides with the one of

$de\sum_{i=0}^{l}\binom{i+r}{r}\binom{l-i+s}{s} = de\binom{l+r+s+1}{r+s+1}$ (this latter purely combinatorial equality can be obtained for instance from (ii) applied to $X = \mathbb{P}^r$, $Y = \mathbb{P}^s$, taking into account that the Hilbert function of a projective space coincides with the Hilbert polynomial for any value of the degree). From this we get that $\deg(J(X,Y)) = de$, which finishes the proof of the proposition. □

Exercise 10.4. Prove that the ideal of $J(X,Y)$ is generated by the elements of $I(X) \cap \mathbb{K}[X_0,\dots,X_m]$ and $I(Y) \cap \mathbb{K}[X_{m+1},\dots,X_n]$.

Remark 10.5. It is not by chance that the proof of the above proposition is so similar to the one of 6.14. In fact, considering X as a projective set in $\mathbb{P}^m$ and Y as a projective set in $\mathbb{P}^{n-m-1}$, the ideal of $J(X,Y)$ coincides as a set with $\overline{I}(X \times Y \subset \mathbb{K}[X_0,\dots,X_n]$, or equivalently $S(J(X,Y))$ coincides with $\overline{S}(X \times Y))$. The difference is that for the join variety we consider the graded structure, while for the product we consider the bigraded structure. In the language of tensor products, $S(X) \otimes S(Y)$ can be endowed naturally with two structures: as a bigraded ring is isomorphic to $\overline{S}(X \times Y))$ (see Remark 6.16), while as a graded ring is isomorphic to $S(J(X,Y))$.

Example 10.6. In the particular case when $Y = V(X_{m+1},\dots,X_n)$, we obtain a cone over the projective set $X \subset \mathbb{P}^m$, which will have the same degree as X. Observe that in this case, the ideal of the cone in $\mathbb{P}^n$ is generated by the equations in $\mathbb{K}[X_0,\dots,X_m]$ defining $I(X)$.

Example 7.17 can be considered as a particular case of join variety (namely when $X = Y$):

Exercise 10.7. If X is a projective variety, prove that the set SX of Example 7.17 is also irreducible, and that it has dimension $2\dim X$ unless X is a linear subspace [Hint: Characterize first a linear subspace as a projective variety such that the line through two general point of it is contained inside the variety]. Conclude that the union of all the lines of SX forms a projective variety of dimension at most $2\dim X + 1$. (The projective variety $S(X)$ is called the *secant variety* of X, and it can be proved that the linear projection of any "reasonable" X from a point $p \in \mathbb{P}^n$ is an isomorphism if and only if $p \notin S(X)$; thus the exercise shows that any "reasonable" projective variety of dimension r can be isomorphically projected into $\mathbb{P}^{2r+1}$, and in general it is not expected that it could be isomorphically projected into $\mathbb{P}^{2r}$.

Exercise 10.8. Prove that the secant variety of a Segre variety, Veronese variety or Grassmannian of lines has always dimension strictly less that the expected one [Hint: show first that a matrix has rank at most two if and only if it can be written as a sum of two matrices of rank one, and find suitable generalizations for symmetric and skew-symmetric matrices].

We see now a nice application of the join variety, which allows to strongly generalize Theorem 7.21.

Proposition 10.9. *Let $X, Y \subset \mathbb{P}^n$ be two projective varieties of respective dimensions r and s. Then any irreducible component of $X \cap Y$ has dimension at least $r+s-n$.*

Proof: We consider X and Y to be projective varieties contained in two skew n-planes inside the same $\mathbb{P}^{2n+1}$ and take their join $J(X,Y) \subset \mathbb{P}^{2n+1}$. Recall that a point $(a_0 : \ldots : a_{2n+1})$ belongs to $J(X,Y)$ if and only if $(a_0 : \ldots : a_n)$ belongs to X and $(a_{n+1} : \ldots : a_{2n+1})$ belongs to Y. Thus we can associate to each point $(a_0 \ldots : a_n)$ in $X \cap Y$ the point $(a_0 : \ldots : a_n : a_0 : \ldots : a_n)$ of $J(X,Y)$. It is immediate to check that this defines an isomorphism between $X \cap Y$ (the intersection regarded in $\mathbb{P}^n$) and $J(X,Y) \cap V(X_0 - X_{n+1}, \ldots, X_n - X_{2n+1})$. Since $J(X,Y)$ is irreducible of dimension $r+s+1$ (from Proposition 10.3(iii)) the proposition follows now from Theorem 7.21 applied to the second intersection. $\square$

In case the dimension of the intersection is the right one, we can say even more.

Proposition 10.10 (Bézout's Theorem)**.** *Let $X, Y \subset \mathbb{P}^n$ be projective varieties of respective dimensions r and s. Then, if $\dim(X \cap Y) = r+s-n$ (i.e. the expected one) the degree of the ideal $I(X)+I(Y)$ is $\deg X \cdot \deg Y$.*

Proof: We keep the same set-up as in the proof of Proposition 10.9, and the idea is to translate that proof into algebra. We know from Proposition 10.3(iii) that $J(X,Y)$ has degree $\deg X \cdot \deg Y$. Moreover, it is clear from Exercise 10.4 that, under the isomorphism $\mathbb{K}[X_0, \ldots, X_{2n+1}]/(X_0 - X_{n+1}, \ldots, X_n - X_{2n+1}) \cong \mathbb{K}[X_0, \ldots, X_n]$, the class of the ideal $I(J(X,Y))$ becomes the ideal $I(X)+I(Y)$. But this means that $I(J(X,Y)) + (X_0 - X_{n+1}, \ldots, X_n - X_{2n+1})$ and $I(X)+I(Y)$ have the same Hilbert polynomials (inside the respective polynomial rings in which they are ideals). In particular, $V(I(J(X,Y)) + (X_0 - X_{n+1}, \ldots, X_n - X_{2n+1}))$ has dimension $r+s-n$. Since $J(X,Y)$ is irreducible of dimension $r+s+1$, this means that the successive intersections with the hyperplanes $V(X_0 - X_{n+1}), \ldots, V(X_n - X_{2n+1})$ have the right dimension, and hence we can apply the weak version of Bézout's Theorem (see Exercise 5.17) to conclude that $I(J(X,Y)) + (X_0 - X_{n+1}, \ldots, X_n - X_{2n+1})$ (and hence $I(X)+I(Y)$) has degree $\deg X \cdot \deg Y$, as wanted. $\square$

Remark 10.11. Proposition 10.3 illustrates a much more general fact. In fact, observe that the map $f : X \times Y \to \mathbb{G}(1,n)$ (whose image is $J_{X,Y}$) is nothing but the Segre map (after the Plücker embedding) when regarding $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^{n-m-1}$. Indeed to each couple $\big((X_0:\ldots:X_m),(Y_{m+1}:\ldots:Y_n)$, the map f associates the line spanned by the rows of the matrix

$$\begin{pmatrix} X_0 & \ldots & X_m & 0 & \ldots & 0 \\ 0 & \ldots & 0 & Y_{m+1} & \ldots & Y_n \end{pmatrix}$$

and hence its Plücker coordinates are either zero or products of the type X_iY_j (all of them in fact). Hence the degree of $J_{X,Y}$ (in the projective space $\mathbb{P}^{\frac{n(n+3)}{2}}$ where $\mathbb{G}(1,n)$ lies in) is $\deg X\cdot\deg Y$ (see Proposition 10.3), which coincides with the degree of $J(X,Y)$ as a subvariety in $\mathbb{P}^n$. Let us justify why this is not casual. Writing $r=\dim X$ and $s=\dim Y$ as in the proof of Proposition 10.3, the degree of $J_{X,Y}$ should be the number of elements in the intersection of $J_{X,Y}$ with $r+s$ "nice" hyperplanes in $\mathbb{P}^{\frac{n(n+3)}{2}}$. But Exercise 1.18 tells us that a possible choice of a hyperplane (unfortunately not the general one) would give, when intersected with $\mathbb{G}(1,n)$, the set of all the lines meeting a given linear space $\Lambda\subset\mathbb{P}^n$ of codimension two. Consider now $r+s$ linear subspaces $\Lambda_1,\ldots,\Lambda_{r+s}\subset\mathbb{P}^n$ of codimension two constructed in the following special way. Fix first $A\subset\mathbb{P}^n$ a linear subspace of dimension $n-r-s-1$ and a hyperplane H containing it. Then we take $\Lambda_1,\ldots,\Lambda_{r+s}\subset\mathbb{P}^n$ to be general hyperplanes in H that contain A (observe that the set of hyperplanes in H with that property form a projective space of dimension $r+s-1$). Consider the hyperplanes $H_1,\ldots,H_{r+s}$ of $\mathbb{P}^{\frac{n(n+3)}{2}}$ whose intersection with $\mathbb{G}(1,n)$ gives the set of all the lines meeting respectively $\Lambda_1,\ldots,\Lambda_{r+s}$. Then it is not difficult to see that a line of $J_{X,Y}$ is in $H_1\cap\ldots\cap H_{r+s}$ if and only if it meets A (although this is not true for an arbitrary line of $\mathbb{P}^n$). But since the degree of $J(X,Y)$ is $\deg X\deg Y$, then a general A should meet $J(X,Y)$ in exactly $\deg X\deg Y$ points, any of each belonging only to one line of $J_{X,Y}$. Hence $J_{X,Y}$ will meet $H_1\cap\ldots\cap H_{r+s}$ in $\deg X\deg Y$ points. This explains why its degree should be $\deg X\deg Y$.

Of course there are two important lacks in the above justification (rather than proof). The first one is that we were often assuming that a variety of degree d "should" meet a general linear space of complementary dimension in exactly d points. This is not a serious objection, since we will see in Theorem 12.1(ii) that this is so. There is however a second objection, which is really serious: the choice of the hyperplanes $H_1\cap\ldots\cap H_{r+s}$ was quite far to be general. I do not know a simple way of proving that this is not a problem (in fact it would be enough to show that the intersection multiplicity will be one). But at least I would like to present some more examples to convince the reader that the use of such special hyperplanes always work fine.

Exercise 10.12. Consider the quadric $Q=V(X_0X_3-X_1X_2)\subset\mathbb{P}^3$. Prove that the set of lines contained in Q is, inside $\mathbb{G}(1,3)\subset\mathbb{P}^5$, the union of two conics.

Exercise 10.13. Generalizing the above exercise, prove that the set of k-planes in $\varphi_{1,k}(\mathbb{P}^1\times\mathbb{P}^k)\subset\mathbb{P}^{2k+1}$, with $k\geq 2$, is a rational normal curve inside some linear subspace of $\mathbb{P}^{\binom{2k+2}{2}-1}$ [Hint: Prove first that any k-plane in the Segre variety has the form $\varphi_{1,k}(\{p\}\times\mathbb{P}^k)$.

Exercise 10.14. Let $C\subset\mathbb{G}(k,n)$ be a curve such that through a general point of $\mathbb{P}^n$ there passes only one k-plane of C. [In fact it can be proved that if through a general point of $\mathbb{P}^n$ there pass more than one line, then C is one of the conics in Exercise 10.12]. Reasoning like in Remark 10.11, justify that the degree of C should coincide with the degree of the ruled variety obtained as the union of all the k-planes parametrized by C.

It is crucial in the above exercise that C is a curve. In fact, the situation for the (special case of) join of two varieties described in Remark 10.11 is very particular, as we see in the next example.

Exercise 10.14. Let $f:\mathbb{P}^2\to\mathbb{G}(1,5)$ be the map that associates to each point $(X_0:X_1:X_2)$ the line in $\mathbb{P}^5$ spanned by the rows of the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 & 0 & 0 & 0\\ 0 & 0 & 0 & X_0 & X_1 & X_2\end{pmatrix}$$

(i) Prove that f is a morphism, and that in fact coincides with the double Veronese embedding into some linear subspace of dimension five in $\mathbb{P}^{14}$ (the projective space containing $\mathbb{G}(1,5)$). Hence f is injective and $Z:=f(\mathbb{P}^2)$ has degree four.

(ii) Prove that the union of the lines of Z coincides with the Segre embedding of $\mathbb{P}^2\times\mathbb{P}^1$, and hence that it has degree three. Prove also that through any point of the Segre variety there passes only one line of Z.

The reason why the degrees do not coincide in the above example is because the right generalization of Exercise 10.14 is the following.

Exercise 10.15. Let $Z\subset\mathbb{G}(k,n)$ be a projective variety of dimension r and degree d after the Plücker embedding. Assume that a general point of a general k-plane of Z is contained in exactly a k-planes of Z. Use the same reasoning as in Remark 10.11 to justify that a times the degree of the union of the k-planes of Z should coincide with the number of k-planes of Z meeting a general linear space of codimension $r+k$.

Let us try to fully explain what is happening in the example of Exercise 10.14. We know that the degree of $Z\subset\mathbb{G}(1,5)\subset\mathbb{P}^{14}$ is four, so that we should find this degree when intersecting with two hyperplanes of $\mathbb{P}^{14}$. We will do it following the philosophy of Remark 10.11. So we take two hyperplanes $H_1,H_2\subset\mathbb{P}^{14}$ whose intersections with $\mathbb{G}(1,5)$ consist of the set of lines in $\mathbb{P}^5$ meeting respectively Λ_1 and Λ_2 (where Λ_1 and Λ_2 are two linear spaces of $\mathbb{P}^5$ of codimension two, which we choose to meet in along a plane Π, or equivalent to span a hyperplane $H\subset\mathbb{P}^5$). Hence $\mathbb{G}(1,5)\cap H_1\cap H_2$ will consist of the set of lines in $\mathbb{P}^5$ either meeting Π or contained in H. The number of lines of Z meeting a general plane Π is by Exercise 10.15 the degree of the union X of the lines of Z, i.e. three (by Exercise 10.14). The rest of the degree comes then from the lines of Z contained in a general hyperplane $H \subset \mathbb{P}^5$, as the following exercise confirms.

Exercise 10.16. With the notation of Exercise 10.14, prove that there is only one line of Z in a general hyperplane of $\mathbb{P}^5$.

Exercise 10.17. Use the previous trick of taking special hyperplane sections to obtain that the degree of $\mathbb{G}(1,4)$ as a projective variety in $\mathbb{P}^9$ is five. [To give an idea of the complexity of trying a general method, the formula for the degree of $G(1,n)$ inside $\mathbb{P}^{\frac{(n-1)(n+2)}{2}}$ is $\frac{1}{n-1}\binom{2n-2}{n}$].

Another question, somehow reciprocal to the one we have been studying in this chapter, would be to study the ruled variety obtained when intersecting a Grassmannian with a series of hypersurfaces. Then from Bézout's theorem (Exercise 5.17) the degree in the projective space where the Grassmannian lies is known (if we know the degree of the Grassmannian, which is not an easy task, as the previous exercise shows).

Example 10.18. The most trivial example consists of intersecting $\mathbb{G}(1,3)$ with the hyperplane $V(p_{01})$ producing the set Z of lines of $\mathbb{P}^3$ meeting the line $V(X_0, X_1)$. Clearly Z has degree two (it is in fact a quadratic cone). But assume now that we only know that Z is the intersection of $\mathbb{G}(1,3)$ with some hypersurface of $\mathbb{P}^5$, and that we want to know the degree of such hypersurface. One way to do it would be to observe that Z has degree two, the same as the degree of $\mathbb{G}(1,3)$, so that the degree of the hypersurface must be one. But we have remarked that in general is not easy to find the degree of a Grassmannian. We try then a geometric and tricky way. The wanted degree will be the number of points in the intersection of the hypersurface with a general line. The trick is now to take the line contained in $\mathbb{G}(1,3)$ (hence again we are not playing fair, since the line is not general), so that the intersection will coincide with the intersection of Z with the line. But a line inside $\mathbb{G}(1,3)$ is a pencil of lines passing through a point p and contained in a plane Π containing p (see Exercise 10.19 below). Then the only line of Z in this pencil is the line spanned by p and the intersection of $V(X_0, X_1,)$ with Π. Therefore this method, even if not fair, gives again the right result.

Exercise 10.19. Let $\Lambda_1, \Lambda_2 \in \mathbb{G}(,k,n) \subset \mathbb{P}^{\binom{n+1}{k}-1}$ be two k-planes in $\mathbb{P}^n$. Prove that the line in $\mathbb{P}^{\binom{n+1}{k}-1}$ generated by Λ_1 and Λ_2 is contained in $\mathbb{G}(k,n)$ if and only if Λ_1 and Λ_2 meet in a linear space A of dimension $k-1$ (or equivalently they span a linear space B of dimension $k+1$). Show that in this case the line in $\mathbb{P}^{\binom{n+1}{k}-1}$ will consist of the set of k-planes containing A and contained in B.

Exercise 10.20. Repeat the computations of Example 10.18 for $V(p_{0\ldots k})$ in any $\mathbb{G}(k,n)$.

11. Tangent spaces and cones; smoothness

Definition. The *embedded projective Zariski tangent space* $\mathbb{T}_xX$ of a projective set $X \subset \mathbb{P}^n$ at a point $x \in X$ is the union of all the lines meeting X at x with multiplicity at least two.

Theorem 11.1. *For any point $x \in X \subset \mathbb{P}^n$, the tangent space $\mathbb{T}_xX$ is a linear space defined by the equations $\frac{\partial F}{\partial X_0}(x)X_0 + \ldots + \frac{\partial F}{\partial X_n}(x)X_n = 0$ as F varies in $I(X)$. Moreover, $\dim \mathbb{T}_xX$ has dimension at least the one of any component of X passing through x.*

Proof: Take a line L meeting X at x and choose coordinates so that $x = (1 : 0 : \ldots : 0)$ and L has equations $X_2 = \ldots = X_n = 0$. If L and X meet at x with multiplicity at least two we can also assume, by Proposition 5.14, that $I(X) \subset (X_1^2, X_2, \ldots, X_n)$. In particular, for any $F \in I(X)$ it holds that $\frac{\partial F}{\partial X_0}(x) = \frac{\partial F}{\partial X_1}(x) = 0$. Hence all the points of L satisfy the equation $\frac{\partial F}{\partial X_0}(x)X_0 + \ldots + \frac{\partial F}{\partial X_n}(x)X_n = 0$.

Reciprocally, assume that all the points of L satisfy the equations $\frac{\partial F}{\partial X_1}(x)X_1 + \ldots + \frac{\partial F}{\partial X_n}(x)X_n = 0$ for any $F \in I(X)$ (observe that $\frac{\partial F}{\partial X_0}(x) = 0$, since $I(X) \subset (X_1, \ldots, X_n)$). This means that $\frac{\partial F}{\partial X_1}(x) = 0$ for all $F \in I(X)$, which is equivalent to say that $F \in (X_1^2, X_2, \ldots, X_n)$. Hence again Proposition 5.14 allows us to conclude that $\mathbb{T}_xX$ is linear and defined by the stated equations (just change coordinates back).

To prove the dimension statement, assume after changing coordinates that $x = (1 : 0 : \ldots : 0)$ and $\mathbb{T}_xX$ has equations $X_{r+1} = \ldots = X_n = 0$ (and hence has dimension r). This means that for each $i = r+1, \ldots, n$ there is a homogeneous polynomial $F_i \in I(X)$ (and hence in particular in $I(Z)$, where Z is any irreducible component of X passing through x) such that $F_i = X_0^{d_i}X_i + G_i$, with $G_i \in (X_1, \ldots, X_n)^2$. Hence we have the following congruences modulo $I(Z) + (X_1, \ldots, X_r)$:

$$\begin{array}{cccc} (X_0^{d_{r+1}} + A_{r+1,r+1})X_{r+1} & + \ldots & +A_{r+1,n}X_n & \equiv 0 \\ \vdots & \ddots & \vdots & \\ A_{n,r+1}X_{r+1} & + \ldots & +(X_0^{d_n} + A_{n,n})X_n & \equiv 0 \end{array}$$

where the $A_{i,j}$'s are homogeneous polynomials in $\mathbb{K}[X_0, X_{r+1}, \ldots, X_n]$ belonging to the ideal $(X_{r+1}, \ldots, X_n)$. Consider the above congruences as a system of linear equations with unknowns $X_{r+1}, \ldots, X_n$. Let B be the matrix of the coefficients of the system, and let F be its determinant. It is then clear that F is a homogeneous polynomial not vanishing at $x = (1 : 0 : \ldots : 0)$. Multiplying this expression with the adjoint matrix of B we get that $FX_{r+1}, \ldots, FX_n \in I(Z) + (X_1, \ldots, X_r)$. Hence $X_1, \ldots, X_n$ are in the homogeneous ideal of any irreducible component of $X \cap V(X_1, \ldots, X_r)$ containing x. This implies that $X \cap V(X_1, \ldots, X_r)$ has only one component passing through x, consisting just of x. By

Theorem 7.21, then Z has dimension at most r, which completes the proof of the theorem. $\square$

Definition. Let x be a point of a projective set X. The *local dimension* of X at x is the maximum dimension of a component of X passing through x. Then x is called a *smooth point* of X if $\dim \mathbb{T}_x X$ coincides with the local dimension of X at x. Otherwise (i.e. if the dimension of $\mathbb{T}_x X$ is bigger) x is called a *singular point* of X.

Proposition 11.2. *Let x be a point of a projective set $X \subset \mathbb{P}^n$ of local dimension r.*

(i) If $I(X)$ is generated by homogeneous polynomials $G_1, \ldots, G_s$ then x is singular if and only if the rank of the matrix $(\frac{\partial G_i}{\partial X_j})_{i=1,\ldots,s;\ j=0,\ldots,n}$ is strictly smaller than $n-r$ at x. In particular, the singular locus of X is a closed subset.

(ii) If X' is the union of the irreducible components of X passing through x, then $\mathbb{T}_x X = \mathbb{T}_x X'$.

(iii) For any irreducible component Z of X, the set of smooth points of X contained in Z is not empty (although valid in any characteristic, we will prove this only when the ground field has characteristic zero).

Proof: It is easy to see that the fact that $I(X)$ is generated by $G_1, \ldots, G_s$ implies that the gradient of any homogeneous polynomial in $I(X)$ is a linear combination of the gradients of $G_1, \ldots, G_s$. Then from the equations of $\mathbb{T}_x X$ found in Theorem 11.1 we immediately obtain (i).

Let X'' be the union of the irreducible components of X not passing through x. We can then take $G \in I(X'') \setminus I(x)$. Therefore $FG \in I(X)$ for any $F \in I(X')$. Hence the gradient at x of any $F \in I(X')$ is a nonzero multiple of the gradient of FG, which implies that $\mathbb{T}_x X' = \mathbb{T}_x X$, i.e. (ii).

In order to prove (iii), observe that from (ii) it is enough to find smooth points of Z not belonging to any other component of X. Hence we need to prove that the intersection in Z of the set of smooth points and the set of points not belonging to other components is not empty. Since Z is irreducible, both sets are open (from (i)) and the second set is not empty, it is enough to prove that the set of smooth points of Z is not empty. Let r be the dimension of Z.

From Proposition 5.7(vi) we can find a linear subspace of codimension $r+1$, which we can assume to be $V(X_0, \ldots, X_r)$, not meeting Z. Therefore we have the morphism $f : Z \to \mathbb{P}^r$ with the properties of Lemma 7.20. In particular for each $i = r+1, \ldots, n$ we can find a monic polynomial $F_i \in \mathbb{K}[X_0, \ldots, X_r, X_i]$ in the variable X_i and belonging to $I(Z)$. We take F_i with minimum degree, say d_i, satisfying this condition. If all the points of Z were singular, then by (i) all the minors of the matrix $(\frac{\partial F_i}{\partial X_j})_{i=r+1,\ldots,n;\ j=0,\ldots,n}$ should be in $I(Z)$. But $\det(\frac{\partial F_i}{\partial X_j})_{i=r+1,\ldots,n;\ j=r+1,\ldots,n} = \frac{\partial F_{r+1}}{\partial X_{r+1}} \cdots \frac{\partial F_n}{\partial X_n}$. Since $I(Z)$ is a prime ideal, then some $\frac{\partial F_i}{\partial X_i}$ belongs to $I(Z)$. But if the characteristic of the field is zero, then $\frac{\partial F_i}{\partial X_i}$ is d_i times a monic polynomial in X_i of degree $d_i - 1$ belonging to $I(Z)$. This contradicts the minimality of d_i and completes the proof of the proposition. □

Exercise 11.3. Prove that, in Examples 1.5, 1.7 and 1.15, a matrix of rank exactly k is a smooth point of the projective variety of matrices of rank at most k. Conclude that Segre and Veronese varieties, as well as Grassmannians of lines are smooth varieties.

Exercise 11.4. Prove that the subset in $\mathbb{P}^{\binom{n+d}{d}-1}$ (the space of all the hypersurfaces of degree d in $\mathbb{P}^n$) corresponding to singular hypersurfaces is a hypersurface of $\mathbb{P}^{\binom{n+d}{d}-1}$.

Exercise 11.5. Prove that the curve in Exercise 1.14 is smooth.

When a point is not smooth we have a better notion than the tangent space. The idea is not to take only linear approximations to our variety at a point, but rather approximations of any degree. This is given by not just the first derivatives, but by the first nonzero derivatives at a point. We start with a general definition.

Definition. Let $F \in \mathbb{K}[X_0, \ldots, X-n]$ be a homogeneous polynomial, $x \in \mathbb{P}^n$ any point and $r \geq 0$ an integer. We define $F_{r,x} = \sum_{i_0+\ldots i_n=r} \frac{\partial^r}{\partial^{i_0}\ldots\partial^{i_n}}(x) X_0^{i_0} \ldots X_n^{i_n}$ (which is defined upto multiplication by a constant). If r is the smallest integer for which $F_{r,x}$ in nonzero, then we define F_x to be $F_{r,x}$. If $I \subset \mathbb{K}[X_0, \ldots, X_n]$ is a homogeneous ideal, we define $C_x I$ to be the ideal generated by all the elements of the form F_x for some homogeneous polynomial $F \in I$.

Lemma 11.6. *Let $I \subset \mathbb{K}[X_0, \ldots, X_n]$ be a homogeneous ideal, and let $x \in \mathbb{P}^n$ be a point.*

(i) $C_x I$ is the whole polynomial ring if and only if $x \notin V(I)$.

(ii) If $x \in X$, $C_x I$ is a homogeneous ideal defining a cone $C_x X \subset \mathbb{P}^n$ with vertex the point x contained in $\mathbb{T}_x X$.

(iii) If G is any homogeneous polynomial, then $C_x(I + (G)) = C_x I + (G_x)$.

Proof: Part (i) is immediate from the definition. Parts (ii) and (iii) are easy exercises when changing coordinates in such a way that x becomes the point of coordinates $(1 : 0 : \ldots : 0)$. In this situation, for any homogeneous polynomial F, we regard it as a polynomial in the variables $X_1, \ldots, X_n$, and F_x is then the nonzero homogeneous part of F with smallest degree (and divided by the maximum possible power of X_0). □

Theorem 11.7. *Let $I \subset \mathbb{K}[X_0, \ldots, X_n]$ be a homogeneous ideal, and let $x \in \mathbb{P}^n$ be a point.*

(i) $V(C_xI)=\{x\}$ *if and only if the only component of* $V(I)$ *passing though* x *is* $\{x\}$.

(ii) If r *is the local dimension on* $V(I)$ *at* x*, then* $V(C_xI)$ *has dimension* r.

Proof: We will assume again that we took coordinates such that $x=(1:0:\ldots:0)$. If $V(I)=\{x\}\cup Y$, with $Y\not\ni x$, we have, by the Nullstellensatz, that $\sqrt{I}=(X_1,\ldots,X_n)\cap I(Y)$, and $I(Y)\not\subset(X_1,\ldots,X_n)$. We can thus find a homogeneous polynomial $G\in I(Y)$ monic in X_0, and also there are powers $(GX_1)^{a_1},\ldots,(GX_n)^{a_n}$ that are in I. But then automatically we have also $X_1^{a_1},\ldots,X_n^{a_n}\in C_xI$, which implies $V(C_xI)=\{x\}$.

For the other implication of (i), we will follow essentially the same steps as in the proof of Theorem 11.1. First, if $V(C_xI)=\{x\}$, using again the Nullstellensatz we know that there is some integer a such that all the monomial of degree a in the variables $X_1,\ldots,X_n$ are in C_xI. Let $M_1,\ldots,M_N$ denote those monomials. By assumption, for each of them, there is a homogeneous polynomial $F_i\in I$ such that $F_{ix}=M_i$ $(i=1,\ldots,N)$. This means that for each $i=1,\ldots,N$ we can write $F_i=X_0^{d_i}M_i+A_{i1}M_1+\ldots+A_{iN}M_N$, with all the A_{ij}'s in the ideal $(X_1,\ldots,X_n)$. As in the proof of Theorem 11.1, we get a system of congruences modulo I

$$\begin{array}{cccc} (X_0^{d_1}-A_{11})M_1- & \ldots & -A_{1N}M_N & \equiv 0\\ \vdots & \ddots & \vdots & \\ -A_{N1}M_1- & \ldots & +(X_0^{d_N}-A_{NN})M_N & \equiv 0 \end{array}$$

that implies $FM_i\in I$ for $i=1,\ldots,N$, where F is the determinant of the coefficients matrix. Since clearly $F(x)\neq 0$, it follows that $M_1,\ldots,M_N$ belong to any primary component of I corresponding to components of $V(I)$ passing through x. Hence there is only one such primary component, and its radical is $(X_1,\ldots,X_n)$.

The proof of (ii) will now follow easily from (i). Let s be the dimension of $V(C_xI)$. We can thus find linear forms $H_1,\ldots,H_s$ vanishing at x such that $V(C_xI)\cap V(H_1,\ldots,H_s)$ is just the point x (since $V(C_xI)$ is a cone of vertex x, identifying the set of lines passing through x with $\mathbb{P}^{n-1}$, we have that $V(C_xI)$ corresponds to a projective set of dimension $s-1$ in $\mathbb{P}^{n-1}$, whose intersection with s general hyperplanes is therefore empty). From Lemma 11.6(iii) we obtain that $C_x(I+(H_1,\ldots,H_s))$ is just the point x. Part (i) implies that also $V\big(I+(H_1,\ldots,H_s)\big)$ has $\{x\}$ as the only irreducible component passing through x. Since $V\big(I+(H_1,\ldots,H_s)\big)=V(I)\cap V(H_1,\ldots,H_s)$, it follows from Theorem 7.21 that any component of X passing through x has dimension at most s. In other words, $r\leq s$, i.e. the local dimension of X at x is at most the dimension of C_xX.

To see the other inequality, we proceed in the same way. We can find r linear forms $H_1,\ldots,H_r$ such that $X\cap V(H_1,\ldots,H_r)$ has only one component passing through x, and this is just $\{x\}$. Using again Lemma 11.6(iii) and part (i) we get that $C_xX\cap V(H_1,\ldots,H_r)$

is just the point x. Theorem 7.21 implies now that $C_x X$ has dimension at most r, completing the proof of (ii). $\square$

Definition. If $X \subset \mathbb{P}^n$ is a projective set and $x \in X$ is a point of it, the cone $V(C_x I(X))$ is called the *tangent cone* of X at x, and it will be denoted by $C_x X$.

12. Transversality

Theorem 12.1. *Let $X \subset \mathbb{P}^n$ be projective variety of dimension r and degree d.*

(i) (Bertini's Theorem) There exists an open subset $U \subset \mathbb{P}^{n*}$ such that, for any $H \in U$ and any smooth point x of X belonging to H, the projective set $X \cap H$ is smooth at x. In particular, if X is smooth or has only a finite number of singular points, the intersection with a general hyperplane is smooth.

(ii) There exists a non-empty open subset $V \subset \mathbb{G}(n-r,n)$ such that, for any linear subspace $\Lambda \in V$, the intersection of X and Λ consists exactly of d different points (each counted with multiplicity one). In particular d is the maximum number of points in the intersection (provided it is finite) of X with a linear space of dimension $n-r$.

Proof: Let X_0 be the open set of X consisting of its smooth points. We consider the incidence variety $I_0 \subset X_0 \times \mathbb{P}^{n*}$ formed by the set of pairs (x,H) for which H contains the tangent space of X at x. We thus know (from Theorem 8.4(iii) together with Lemma 7.8(iii)) that I_0 is irreducible of dimension $r+(n-r-1)=n-1$ (because the fibers of the projection onto X_0 are projective spaces of dimension $n-r-1$, corresponding to the set of hyperplanes containing the tangent space at a point). Taking the Zariski closure of I_0 in $X \times \mathbb{P}^{n*}$ we get a projective variety of dimension $n-1$. Therefore its projection to $\mathbb{P}^{n*}$ is a proper closed subset. The complement of this closed set is the wanted open set U. Indeed, for any smooth point x of X belonging to H, it is clear that the tangent space to $X \cap H$ at x must be contained in both $\mathbb{T}_x X$ and H. Since $\mathbb{T}_x X \not\subset H$, then $\dim(\mathbb{T}_x X \cap H) \leq r-1$. But then Proposition 11.2(i) implies that x is a smooth point of $X \cap H$, proving the first part of (i). For the second part, it is just enough to intersect the above open set with the open set of hyperplanes not passing through any singular point (this last open set is not empty when the number of singular points is finite).

For the proof of (ii), we already know from Exercise 7.19 that we have an open set in $\mathbb{G}(n-r,n)$ of linear spaces whose intersection with X has dimension zero, i.e. is a finite number of points. Hence we know from Theorem 5.15 that this number of points, counted with multiplicity, is d. So that we just need to prove that we can find a smaller open set in which the intersection multiplicity is always one. By Proposition 5.14, a linear space Λ will meet X at x with multiplicity at least two if and only if Λ and $\mathbb{T}_x X$ share at least one line passing through x. So we will just imitate the proof of part (i). We consider now the incidence variety $J_0 \subset X_0 \times \mathbb{G}(n-r,n)$ of pairs (x,Λ), where x is a smooth point of X and Λ a linear space meeting $\mathbb{T}_x X$ in at least one line through x. We now use the projection of J_0 onto X_0 to obtain the dimension of J_0. If we fix a smooth point x of X, its fiber under this projection consists of the set of $(n-r)$-planes meeting $\mathbb{T}_x X$ along a line passing through x. Identifying the set of $(n-r)$-planes passing through x with $\mathbb{G}(n-r-1, n-1)$ (see Exercise 7.7) the fiber is then isomorphic to the set of $(n-r-1)$-planes in some $\mathbb{P}^{n-1}$ meeting a fixed linear space of dimension $r-1$. From Exercise 1.18 this is a hyperplane section of $\mathbb{G}(n-r-1, n-1)$. Therefore J_0 is irreducible of dimension $r + \dim \mathbb{G}(n-r-1, n-1) - 1 = \dim \mathbb{G}(n-r, n) - 1$. Hence the Zariski closure of J is a projective set in $X \times \mathbb{G}(n-r, n)$ whose image under the second projection is necessarily a proper closed subset of $\mathbb{G}(n-r, n)$. The complement of this closed subset is not still the wanted open set, since after taking closures we can still find singular points. But another easy dimension counting (see Exercise 8.9) shows that the set of $(n-r)$-planes meeting the singular locus (which has dimension at most $r-1$ by Proposition 11.2) is another proper closed subset of $\mathbb{G}(n-r, n)$. The proof of the Theorem is now complete since we can take the complement of the union of the proper closed sets that we have found. $\square$

Corollary 12.2. *Let $X \subset \mathbb{P}^n$ be a projective variety of degree d and dimension $r \leq n-2$. Then there exists a non-empty open subset $U \subset \mathbb{P}^n$ such that for any $p \in U$ the image of X in $\mathbb{P}^{n-1}$ under the linear projection from p has degree d. In particular, the cone over X with vertex a general point has degree d.*

Proof: We leave as an exercise to prove that the set of points p for which the image of X has degree different from d is a closed set. So we just prove that there is good center of projection p. For this purpose, take a linear subspace Λ (whose existence was proved in Theorem 12.1(ii)) of dimension $n-r$ meeting X exactly in d points. Take p to be a point in Λ not contained in any line spanned by two out of the d points of $X \cap \Lambda$ (this p exists since $n-r \geq 2$). Therefore the projection from p yields a variety $\overline{X} \subset \mathbb{P}^{n-1}$ of dimension r and such that the intersection with the $(n-r-1)$-plane image of Λ consists of exactly d different points. Since clearly the intersection of $\overline{X}$ with any other $(n-r-1)$- plane, if finite, consists of at most d points, it follows again from Theorem 12.1(ii) that $\deg \overline{X} = d$, as wanted. $\square$

Exercise 12.3. Let $Z \subset \mathbb{G}(n-r-1, n)$ be the projective variety (of codimension one) consisting of $(n-r-1)$-planes meeting a projective variety $X \subset \mathbb{P}^n$ of dimension r and degree d (see Exercise 8.9). Use Corollary 12.2 and the method of Example 10.18 to conclude that, if Z is the intersection of $\mathbb{G}(n-r-1, n)$ with a hypersurface, then this hypersurface has degree d. [The hypothesis that Z is such an intersection is not restrictive at all, since it can be proved that any subvariety of codimension one of any Grassmannian is obtained as the intersection of the Grassmannian with some hypersurface].

Exercise 12.4. Prove that any projective variety of degree one is a linear subspace.

Exercise 12.5. Let $X \subset \mathbb{P}^n$ be a smooth projective variety and dimension r.

(i) Prove that the set consisting of the tangent spaces at all the points of X forms a projective variety in $\mathbb{G}(k, n)$.

(ii) Describe the above variety when X is the twisted cubic in $\mathbb{P}^3$.

(iii) More generally, if $k \geq r$, prove that the union of all the k-planes of $\mathbb{P}^n$ containing some tangent space of X is a projective variety in $\mathbb{G}(k, n)$ (when $k = n-1$, the corresponding variety $X^* \subset \mathbb{P}^{n*}$ is called the *dual variety* of X).

(iv) Show that the "expected dimension" of X is $n-1$. Prove that indeed the dual variety of a smooth curve is a hyperplane if and only if the curve is not contained in any hyperplane.

(v) Find the equation of the dual variety of the twisted cubic. What is its singular locus?

One important application of the description of the degree given in Theorem 12.1(ii) is the characterization of curves of minimal degree.

Theorem 12.6. *Let $X \subset \mathbb{P}^n$ be an irreducible curve of degree d that it is not contained in a hyperplane. Then $d \geq n$, and equality holds if and only if X is a rational normal curve.*

Proof: The first part is easy. Just take n different points of X. They are contained in a hyperplane $H \subset \mathbb{P}^n$. Since $X \not\subset \mathbb{P}^n$ and X is irreducible, the intersection of X and H must consist of d points counted with multiplicities. Since there are at least n different points in the intersection, it follows that $d \geq n$.

Assume now $d = n$. Fix $n-1$ points $p_1, \dots, p_{n-1}$ of X and let $\Lambda \subset \mathbb{P}^n$ be their linear span. By the same reason as above, Λ has dimension $n-2$ since X is not contained in a hyperplane. Let us consider the pencil of hyperplanes containing Λ. As a subset of $\mathbb{P}^{n*}$, this is a projective line, so that it can be identified with $\mathbb{P}^1$. For each element H of the pencil, we know that H meets X in n points counted with multiplicity. And in fact the general H meets X in a point different from $p_1, \dots, p_{n-1}$ (we know from Theorem 12.1 that the elements of the pencil for which we get n different points is an open set of $\mathbb{P}^1$; and on the other hand, taking $p_n \neq p_1, \dots, p_{n-1}$ we get a hyperplane in that open set). We therefore get a morphism from an open set $U \subset \mathbb{P}^1$ to X, associating to each element H of U the intersection point of X and H outside $\{p_1, \dots, p_{n-1}\}$. But then Proposition 8.3 implies that this morphism extends to a morphism $f : \mathbb{P}^1 \to X$.

By Proposition 8.1, f is thus defined by homogeneous polynomials $F_0, \dots, F_n \in \mathbb{K}[T_0, T_1]$ of the same degree. It is very reasonable to think that the degree of these polynomials, say m, must be n, since any hyperplane in $\mathbb{P}^n$ corresponds to a linear combination of $F_0, \dots, F_n$, which is still a homogeneous polynomial of degree m. Since any root of such a polynomial provides a point of X meeting the hyperplane, and a homogeneous polynomial of degree m in $\mathbb{K}[T_0, T_1]$ has m roots counted with multiplicity, m must be the degree of X, i.e. n. However, we do not know that the multiplicity of a root corresponds with the intersection multiplicity of X and the hyperplane at the corresponding point. The idea will be thus to show that we can find a homogeneous polynomial in the linear span of $F_0, \ldots, F_n$ having only simple roots.

Assume for a while that we proved that $m = n$. Then $F_0, \ldots, F_n$ would form a basis of the space of homogeneous polynomials of degree n (they cannot be linearly dependent over $\mathbb{K}$, since this would imply that X is contained in a hyperplane). Therefore after a linear change of coordinates in $\mathbb{P}^n$ we would obtain the curve defined by the set of all the monomials of degree n (which form another basis of this space of polynomials). Therefore X would be a rational normal curve, which is what we want to prove.

So let us prove $m = n$. Let V be the linear span in $\mathbb{K}[T_0, T_1]_m$ of $F_0, \ldots, F_n$. I claim that it is possible to find two polynomials $F, G \in V$ without a common factor. Indeed we pick F to be any element of V (although later on we will need to be more careful choosing F). And now, for each of the linear factors of F, the set of polynomials of V divisible by this factor is a hyperplane in V (recall that $F_0, \ldots, F_n$ do not share a common factor). It is thus enough to take G in the complement of this finite set of hyperplanes of V. We want to prove that there exists some $\lambda \in \mathbb{K}$ for which $P_\lambda := F + \lambda G$ has m different roots. We know that P_λ has repeated roots if and only if its discriminant is zero. But its discriminant in a polynomial in $\mathbb{K}[\lambda]$. So if any P_λ has multiple roots then this discriminant is zero. But this implies that P_λ, as a polynomial with coefficients in the field $\mathbb{K}(\lambda)$, has also a multiple root. This means that we can write $P_\lambda = Q(\lambda, T_0, T_1)^2 R(\lambda, T_0, T_1)$, with $Q, R \in \mathbb{K}(\lambda)[T_0, T_1]$ and being homogeneous of positive degree in T_0, T_1. But removing denominators it is easy to see that we can assume that Q and R are also polynomical in in λ. But since P_λ has degree one in λ, then Q must have degree zero, i.e. do not depend on λ. Hence we find that $Q \in \mathbb{K}[T_0, T_1]$ is a common factor for all the polynomials $F + \lambda G$ when λ varies in $\mathbb{K}$, which is impossible.

We therefore can find a polynomial in V having m different roots in $\mathbb{P}^1$. But this is not still sufficient to conclude, since two of its roots might yield the same point of X. The way of solving this is to be smarter when choosing F above. Observe first that the map $f : \mathbb{P}^1 \to X$ is injective in an open set of $\mathbb{P}^1$. This means that the set of points of $\mathbb{P}^1$ for which there exists a "partner" sharing the same image in X is finite. We can therefore find a polynomial $F \in V$ not vanishing at any of this finite number of points. Therefore, only for a finite number of values of λ above we will have that $F + \lambda G$ vanishes at some of those points. On the other hand, also for a finite number of values of λ we get that the discriminant of $F + \lambda G$ is zero. Taking λ outside both finite sets, we get a polynomial in V (which corresponds to a hyperplane H of $\mathbb{P}^n$) with m different roots, each yielding a different point in the intersection of X and H. This means that the degree of X is at least m, i.e. $m \geq n$. Since clearly $m \leq n$ (otherwise $F_0, \ldots, F_n$ would be linearly dependent and X would be contained in a hyperplane) it follows $m = n$, which completes the proof of the theorem. □

An important application of the above theorem is the following result on the arithmetic genus of curves.

Theorem 12.7. *Let $X \subset \mathbb{P}^n$ be an irreducible curve. Then its arithmetic genus is non-negative and it is zero if and only if X is isomorphic to $\mathbb{P}^1$.*

Proof: Let $m \in \mathbb{Z}$ be big enough so that for $l \geq m$ it holds $\dim_{\mathbb{K}} S(X)_l = P_X(l)$. If d is the degree of X, then $\dim_{\mathbb{K}} S(X)_l = dl + 1 - p_a(X)$ if $l \geq m$. We now consider the m-uple Veronese embedding ν_m of $\mathbb{P}^n$. Since $\dim_{\mathbb{K}} S(X)_m = dm+1-p_a(X)$, it follows that $\nu_m(X)$ spans a linear space of dimension $N := dm - p_a(X)$. We can therefore consider $\nu_m(X)$ as an irreducible curve in $\mathbb{P}^N$ not contained in any hyperplane. Since $\dim_{\mathbb{K}} S(X)_{ml} = dlm + 1 - p_a(X)$ for each $l \geq 1$, it follows that $\dim_{\mathbb{K}} S(\nu_m(X))_l = dml + 1 - p_a(X)$. Therefore, $\nu_m(X)$ has degree dm. But now Theorem 12.6 implies $dm \geq N$, or equivalently $p_a(X) \geq 0$.

On the other hand, if equality holds, then $\nu_m(X)$ must be a rational normal curve, and hence it is isomorphic to $\mathbb{P}^1$. Since ν_m is also an isomorphism, it follows that X is also isomorphic to $\mathbb{P}^1$. Reciprocally, if X is isomorphic to $\mathbb{P}^1$, then by Theorem 8.15 it holds $p_a(X) = p_a(\mathbb{P}^1) = 0$. This proves the theorem. □

13. Parameter spaces

The Grassmannian of k-planes in $\mathbb{P}^n$ is a nice projective variety parametrizing a very particular type of projective sets (as shown in Exercise 12.4, they are exactly those of dimension k and degree one). The scope of this chapter is to explain (avoiding the deep technicalities) how it is possible to find nice parameter spaces for all the projective sets with the same invariants. Since the invariants of a projective set are codified in the Hilbert polynomial, a natural possibility is to try to give an algebraic structure to the set of projective sets with the same Hilbert polynomial. But on the other hand, the example of $\mathbb{G}(k,n)$ seems to show that it is enough to consider the dimension and the degree. In fact both approaches will be valid, and will give rise respectively to the Hilbert schemes and to the Chow varieties, which we will study here.

The main idea to study Hilbert schemes is very simple. Assume we fix a polynomial $P \in \mathbb{Q}[T]$ for which we want to find the set of all the projective sets in $\mathbb{P}^n$ having P as their Hilbert polynomial. If $X \subset \mathbb{P}^n$ has Hilbert polynomial $P_X = P$, then we know that there exists $l_0 \in \mathbb{N}$ such that $\dim(S(X)_l) = P(l)$ if $l \geq l_0$. In particular this implies that for $l \geq l_0$ we know exactly the dimension of the subspace $I(X)_l$ of $S_l = \mathbb{K}[X_0, \ldots, X_n]_l$. Moreover, for l big enough (for instance at least the maximum degree of the elements of a set of generators for $I(X)$) the homogeneous part of degree l determines univoquely the projective set X (it is not difficult to see that $l \geq \deg X$ is enough for this purpose). Projectivizing, we can associate to each projective set X the linear subspace $\mathbb{P}(I(X)_l)$ of $\mathbb{P}(S_l)$, i.e. an element of a Grassmannian. The first problem is that l_0 depends on each particular X, so that it could happen that for each l we can find a projective set X with Hilbert polynomial P and such that $\mathbb{P}(I(X)_l)$ does not have the dimension predicted by P. Fortunately it is not so. For instance, we have seen in Proposition 3.8 that when we are dealing with d points (i.e. when $P = d$) we can take $l_0 = d - 1$. In general the situation is similar, and we have the following theorem (which we will not prove).

Theorem 13.1. *Let $P \in \mathbb{Q}[T]$ a polynomial and fix $n \in \mathbb{N}$. Then there exists an integer l_0 such that for any projective set $X \in \mathbb{P}^n$ and any $l \geq l_0$ it holds that* $\dim(S(X)_l) = P(l)$ *and* $V(I(X)_l) = X$.

Proof: See for instance [Se]. □

The above theorem then shows that, writing $n(l) = \dim(S_l) - 1$ and $k(l) = n(l) - P(l)$, we have for any $l \geq l_0$ an injective map that associates to any projective set $X \subset \mathbb{P}^n$ with $P_X = P$ an element of $\mathbb{G}(k(l), n(l))$. Let us study this map in a very concrete and simple example.

Example 13.2. Consider the case $n = 1$ and $P = 2$, (i.e. sets of two points in $\mathbb{P}^1$). As just remarked, Proposition 3.8 tells us that $h_X(1) = 2$ for any set X of two points. But then $I(X)_1$ is zero, and thus it does not determine X. But clearly X is always determined $I(X)_2$ (since in fact $I(X)$ is generated by a homogeneous polynomial of degree two), so that we can take $l = 2$. Thus we have an injective map that associates to any set $X \subset \mathbb{P}^1$ consisting of two points the element of $\mathbb{G}(0, 2) = \mathbb{P}(\mathbb{K}[X_0, X_1]_2)$ corresponding to the generator (up to a constant) of $I(X)$. But the image of this map is not the whole $\mathbb{P}(\mathbb{K}[X_0, X_1]_2)$, since a homogenous polynomial of degree two determines two points if and only if its discriminant is not zero. Therefore the set of pairs of points in $\mathbb{P}^1$ is parametrized by an open set of the projective plane $\mathbb{P}(\mathbb{K}[X_0, X_1]_2)$. Once again, we would need to also consider infinitely close points in order to obtain a projective parameter space.

Exercise 13.3. Prove that sets of d points in $\mathbb{P}^1$ are parametrized by an open set of $\mathbb{P}^d$.

Example 13.4. It is not always the case that the subset in $\mathbb{G}(k(l), n(l))$ parametrizing projective sets with fixed Hilbert polynomial is dense in that Grassmannian. Consider now for instance $n = d = 2$, i.e. pairs of points in $\mathbb{P}^2$. As in the case of $\mathbb{P}^1$, it is easy to see that we can take $l = 2$, so that to any set $X \subset \mathbb{P}^2$ of two points we associate an element of $\mathbb{G}(3, 5)$, namely the one corresponding to the linear system of conics passing through the two points. It is clear by just dimensional reasons (pairs of points in $\mathbb{P}^2$ must depend on four parameters, while $\mathbb{G}(3, 5)$ has dimension eight) that the image of the set of pairs of points is not dense in $\mathbb{G}(3, 5)$. But it is also clear to see it directly, since a general web of conics (i.e. a general element of $\mathbb{G}(3, 5)$) defines the empty set, and only very particular ones determine two points. In fact a necessary condition is that the web contains the elements X_0L, X_1L, X_2L, where L is the equation of the line passing through the points of X. If, as in the previous example, we also consider infinitely close points, the condition is also sufficient since then the two points are defined as the intersection of the line $V(L)$ with the conic defined by any element of the web not divisible by L. We have then shown that the parameter space we are looking for is the image under the second projection of the incidence variety $I \subset \mathbb{P}^{2^*} \times \mathbb{G}(3, 5)$ consisting of the pairs $(V(L), A)$ where A is a web of conics such that $X_0L, X_1L, X_2L \in A$. Using Theorem 8.4(iii) for the second projection the reader can easily verify that we get that I is a projective variety of dimension four, and that hence the parameter space of pairs of points in $\mathbb{P}^2$ is a projective variety of dimension four.

Exercise 13.5. Imitate the above example to construct a space parametrizing conics in $\mathbb{P}^3$ (i.e. projective sets contained in a plane and defined by a quadric equation inside the plane). Show that this set is a projective variety when considering double lines in a plane as conics. As in Exercise 8.6, study for which degrees d it holds that any surface in $\mathbb{P}^3$ of degree d contains necessarily a conic.

The above examples are quite representative of the general situation. The general parameter space of projective sets with fixed Hilbert polynomial will be, when we allow degenerations, a projective set inside $\mathbb{G}(k(l), n(l))$ (not necessarily irreducible).

Theorem 13.6. *Let $P \in \mathbb{Q}[T]$ a polynomial and $n \in \mathbb{N}$ a fixed integer. Then there exists a projective set $H_P(\mathbb{P}^n) \subset \mathbb{G}(k(l), n(l))$ (with the notation above) and an open set $H'_P(\mathbb{P}^n) \subset H_P(\mathbb{P}^n)$ such that the ideal corresponding to any element of $H'_P(\mathbb{P}^n)$ has Hilbert polynomial P.*

Proof: We would like to identify inside $Z \subset \mathbb{P}^n \times \mathbb{G}(k(l), n(l))$ the subset consisting of pairs $(x, \mathbb{P}(A))$ such that the ideal I_A generated by A has Hilbert polynomial P and $x \in V(I_A)$.

For any $\mathbb{P}(A) \in \mathbb{G}(k(l), n(l))$ (where A is a linear subspace of S_l), write I_A for the ideal generated for A. Fix a set of generators G of A as a vector space over $\mathbb{K}$ (recall from the description given for the equations of the Grassmannians that we can find generators having as coordinates Plücker coordinates). In order to impose that the Hilbert function of I_A is defined by P for $l' \geq l$ we just need to check that, for each $e \geq 0$, the set of products of a monomial of degree e by an element of G spans inside S_{l+e} a linear space of dimension $\binom{n+l+e}{l+e} - P(l+e)$. What is a closed condition inside $\mathbb{G}(k(l), n(l))$ is to impose that this dimension is at most $\binom{n+l+e}{l+e} - P(l+e)$. Intersecting all these closed sets we get a projective set in $H_P(\mathbb{P}^n) \subset \mathbb{G}(k(l), n(l))$ with the property that for any $\mathbb{P}(A)$ in $H_P(\mathbb{P}^n)$ it it holds that $\dim(S/I_A)_{l'} \geq P(l')$ for any $l' \geq l$.

To see that the subset $H'_P(\mathbb{P}^n) \subset H_P(\mathbb{P}^n)$ for which the above inequalities are equalities is in fact an open set we need to work a little bit more, but it is based on an easy observation. First of all, consider the second projection map from the incidence variety in $\mathbb{P}^n \times \mathbb{G}(k(l), n(l))$ of pairs $(x, \mathbb{P}(A))$ for which $x \in I_A$. Thus we see that the subset of $\mathbb{G}(k(l), n(l))$ of subspaces $\mathbb{P}(A)$ for which $V(I_A)$ has dimension exactly $r := \deg P$ is locally closed in $\mathbb{G}(k(l), n(l))$. Hence the subset of $H_P(\mathbb{P}^n)$ of subspaces for which $\dim V(I_A) = r$ is open. But the key observation is that then subspaces on this open set define ideals with Hilbert polynomial of degree r, and thus univoquely determined by just $r+1$ values. In other words, in the intersection of the $r+1$ open sets defined respectively by the conditions $\dim(S/I_A)_{l'} = P(l')$ for $l' = l, l+1, \ldots, l+r$, then we get that automatically $\dim(S/I_A)_{l'} = P(l')$ for any $l' \geq l$. Hence $H'_P(\mathbb{P}^n)$ is the intersection of a finite number of open sets, so that it is open itself, just finishing the proof of the theorem. □

We now introduce the other notion of parameter space, namely the Chow variety. The main idea is to define any projective variety with just one equation in some other space. This is done in the following way. Assume $X \subset \mathbb{P}^n$ is a projective variety of dimension r and degree d. Then Exercise 8.9 implies that the set of linear spaces of dimension $n-r-1$ meeting X is a hypersurface in $\mathbb{G}(n-r-1,n)$. It can be proved (but it is not trivial at all) that any hypersurface of a Grassmannian $\mathbb{G}(n-r-1,n)$ is the intersection of the Grassmannian with a hypersurface in $\mathbb{P}^{\binom{n+1}{r+1}-1}$, the projective space in which $\mathbb{G}(n-r-1,n)$ lies under the Plücker embedding. Moreover, the degree of this hypersurface coincides with the degree of X (see Exercise 12.3). Therefore X is represented by an element of $\mathbb{P}(S(\mathbb{G}(n-r-1,n))_d)$.

Exercise 13.7. Let $X\subset\mathbb{P}^3$ be a plane curve of degree d defined by the equations $X_3=0$ and $F=0$, where $F\in\mathbb{K}[X_0,X_1,X_2]$ is a homogeneous polynomial of degree d. Find in a homogeneous polynomial $G\in\mathbb{K}[p_{01},p_{02},p_{03},p_{12},p_{13},p_{23}]$ of degree d such that the $V(G)\cap\mathbb{G}(1,3)$ is the set of all the lines of $\mathbb{P}^3$ meeting X.

However in order to avoid the use of a nontrivial result such as the one above we will use the same philosophy, but replacing $\mathbb{G}(n-r-1,n)$ with sets of $n-r$ points generating it. This would yield to an alternative construction of a Chow variety. The reader is invited to produce the parallel construction of Chow variety arising from the use of the Grassmannian (which is in fact much more natural). Observe that an equivalent way to ours would be to replace $\mathbb{G}(n-r-1,n)$ with sets of $r+1$ hyperplanes defining it, as it is done in [H].

We thus consider U to be the open set of the product $\mathbb{P}^n\times\overset{n-r)}{\cdots}\times\mathbb{P}^n$ consisting of the sets of points whose linear span has dimension $n-r-1$. It is an easy exercise to prove that the subset of U consisting of sets of points whose linear span meets X is a hypersurface in U. Therefore, its closure in $\mathbb{P}^n\times\overset{n-r)}{\cdots}\times\mathbb{P}^n$ is defined by just one equation $F_X\in\mathbb{K}[X_{10},\dots,X_{1n},\dots,X_{n-r-1,0},\dots,X_{n-r-1,n}]$ (as it was proved in Proposition 5.9 in the homogeneous case). Let us see that its multidegree is $(d,\dots,d)$. So fix $n-r-1$ general points $p_1,\dots,p_{n-r-1}\in\mathbb{P}^n$. When we substitute in F_X $n-r-1$ sets of variables by the coordinates of $p_1,\dots,p_{n-r-1}$ (the order is not important) we obtain a homogeneous polynomial $G\in\mathbb{K}[X_0,\dots,X_n]$, which we want to prove to have degree d. The polynomial G vanishes on those points whose span with $p_1,\dots,p_{n-r-1}$ meets X. In other words, G is the equation of the cone over X with vertex the linear span of the points $p_1,\dots,p_{n-r-1}$. Since the points are general, the degree of this cone is d (by Corollary 12.2), as wanted. On the other hand, it is not difficult to see that F_X must be irreducible, since $V(F_X)$ is irreducible (because X is).

Definition. The multihomogeneous irreducible polynomial F_X produced by the above construction is called *the Chow form* of X.

Exercise 13.8. Find the Chow form of the curve of Exercise 13.7.

We therefore found a method to associate to each projective variety of any dimension r a unique equation $F_X\in\mathbb{K}[X_{10},\dots,X_{1n},\dots,X_{n-r-1,0},\dots,X_{n-r-1,n}]$ (unique up to multiplication by a constant). We can then represent any projective variety of dimension r

and degree d by a point in the projective space $\mathbb{P}$ defined by the multihomogeneous polynomials of multidegree $(d, \ldots, d)$ in $\mathbb{K}[X_{10}, \ldots, X_{1n}, \ldots, X_{n-r-1,0}, \ldots, X_{n-r-1,n}]$. Our next goal is to see that the set of all the elements of $\mathbb{P}$ corresponding to Chow forms has a nice algebraic structure.

Theorem 13.9. *Let $Ch_{r,d}(\mathbb{P}^n) \subset \mathbb{P}$ the set of all Chow forms of varieties of dimension r and degree d.*

(i) The assignment $X \mapsto [F_X] \in Ch_{r,d}(\mathbb{P}^n)$ is one-to-one.

(ii) $Ch_{r,d}(\mathbb{P}^n)$ is a quasiprojective set.

Proof: Given a Chow form F_X, the intersection of $V(F_X)$ with the open set $U \subset \mathbb{P}^n \times \overset{n-r)}{\cdots} \times \mathbb{P}^n$ defined above consists of those sets of $n-r$ points whose linear span has dimension $n-r-1$ and meets X. Obviously, a point $p \in \mathbb{P}^n$ is in X if and only if, for any $(n-r-1)$-plane passing through p, any choice of $n-r$ points spanning it defines a point of $V(F_X) \cap U$. This immediately proves (i).

Following with the same idea, given any $F \in \mathbb{P}$, we consider again $V(F) \cap U$. In order to see whether $F = F_X$ we need to find the points $p \in \mathbb{P}^n$ for which for any $(n-r-1)$-plane passing through p and any choice of $n-r$ points spanning it defines a point of $V(F) \cap U$. In other words, consider first the incidence variety $I_F \subset \mathbb{P}^n \times \big(V(F) \cap U\big)$ consisting of elements $(p, p_1, \ldots, p_{n-r})$ such that $p \in < p_1, \ldots, p_{n-r} >$ and let π be the projection to the first factor. We are then interested in the points of $\mathbb{P}^n$ for which the fiber of π has maximum dimension, i.e. $(n-r-1)(r+1) + (n-r)(n-r-1) = (n+1)(n-r-1)$. This is then a closed subset $X \subset \mathbb{P}^n$ by Proposition 7.18(iii) (if you want to use it directly you would need to take the Zariski closure of I_F inside $\mathbb{P}^n \times (\mathbb{P}^n \times \overset{n-r)}{\cdots} \times \mathbb{P}^n)$. We want to know when $F = F_X$. We would then need at least X to be a projective variety of dimension r and degree d.

I claim that in fact it is enough to know that X has dimension r. Indeed as soon as X has a component Y of dimension r, then $V(F)$ contains contains $V(F_Y)$ as a component. Therefore F is divisible by F_Y. If we assume F to be irreducible (which is possible if we restrict ourselves to an open set of $\mathbb{P}$) then $F = F_Y$, from which necessarily Y is the whole X and has degree d, as claimed.

So the above claim suggests us to consider the open set $V \subset \mathbb{P}$ of irreducible hypersurfaces and consider the incidence set $I \subset V \times \mathbb{P}^n \times U$ of elements $\big([F], p, (p_1, \ldots, p_{n-r})\big)$ for which $(p_1, \ldots, p_{n-r}) \in V(F)$ and $p \in < p_1, \ldots, p_{n-r} >$ (i.e. $\big(p, (p_1, \ldots, p_{n-r})\big) \in I_F$). Considering the projection $q : I \to \mathbb{P}$, $Ch_{r,d}(\mathbb{P}^n)$ consists of the points of V for which the fiber over q has dimension r, so it is a quasiprojective set, as wanted. $\square$

Remark 13.10. We have constructed the above Chow variety $Ch_{r,d}(\mathbb{P}^n)$ (by the way the name is not very appropriate, since it is seldom irreducible) for irreducible varieties X. In order to also allow reducible varieties (but with all components of dimension r), we should also consider the images in $\mathbb{P}$ of the different products $Ch_{r,d_1}(\mathbb{P}^n) \times \ldots \times Ch_{r,d_s}(\mathbb{P}^n)$ with $d_1 + \ldots + d_s = d$ (given by the multiplication of the corresponding Chow forms). But they are still quasiprojective sets.

Definition. Abusing the notation, we will still denote by $Ch_{r,d}(\mathbb{P}^n)$ the quasiprojective set (called *Chow variety*) parametrizing projective sets in $\mathbb{P}^n$ of pure dimension r and degree d.

Even if it is conceptually simpler, the Chow variety is not as good as the Hilbert scheme. For instance, in $Ch_{1,2}(\mathbb{P}^3)$, we would obtain pairs of meeting lines as a degeneration of smooth conics. But we know that this is not a good degeneration, since the arithmetic genus is not the same. In fact, Chow varieties cannot "see" components of smaller dimension (in particular embedded components). And this is why they do not behave well under degenerations.

14. Affine varieties vs projective varieties; sheaves

So far we have essentially concentrated on projective varieties instead affine varieties. The reason was that global properties (degree, intersection results like Bézout's theorem,...) only hold for projective varieties, since they are "complete", while affine varieties have their points at infinity missing. However, there are some properties behaving well for affine varieties (for instanced those regarding morphisms), even much better than in the projective case. We want to analyze in this section some good properties of affine varieties, which will be in fact crucial in order to define new concepts regarding projective varieties (or a generalization of them). We start from the basic definitions.

Definition. An *affine set* in $\mathbb{A}^n$ is a subset defined as the zero locus $V(T)$ of a subset of polynomials $T \subset \mathbb{K}[X_1, \dots, X_n]$. Reciprocally, given a subset $X \subset \mathbb{A}^n$ we defined its *affine ideal* $I_a(X)$ to be the set of all the polynomials of $\mathbb{K}[X_1, \dots, X_n]$ vanishing at all the points of X.

We have essentially the same properties for the operators V and I_a that we had in the projective case (sometimes the situation is easier since we should not worry about homogeneity). We collect them as an exercise.

Exercise 14.1. Prove the following properties:

(i) $I_a(\mathbb{A}^n) = \{0\}$ (again for this we just need $\mathbb{K}$ to be infinite) $V(\{0\}) = \mathbb{A}^n$, and $V(\{1\}) = \emptyset$.

(ii) If $T \subset \mathbb{K}[X_1, \dots, X_n]$ and $< T >$ is the ideal generated by T, then $V(T) = V(< T >)$. In particular, any affine set can be defined by a finite number of equations.

(iii) If $T \subset T' \subset \mathbb{K}[X_1, \dots, X_n]$, then $V(T') \subset V(T) \subset \mathbb{P}^n$.

(iv) If $\{T_j\}_{j\in J}$ is a collection of subsets of $\mathbb{K}[X_1, \dots, X_n]$ then $V(\bigcup_{j\in J} T_j) = \bigcap_{j\in J} V(T_j)$.

(v) If $\{I_j\}_{j\in J}$ is a collection of ideals of $\mathbb{K}[X_1, \dots, X_n]$ then $V(\sum_{j\in J} I_j) = \bigcap_{j\in J} V(I_j)$.

(vi) If $I \subset \mathbb{K}[X_1, \dots, X_n]$ is any ideal, then $V(I) = V(\sqrt{I})$.

(vii) If $I, I' \subset \mathbb{K}[X_1, \dots, X_n]$ are two ideals, then $V(I \cap I') = V(II') = V(I) \cup V(I')$.

(viii) For any $X \subset \mathbb{A}^n$, $I_a(X)$ is a radical ideal. If X is an affine set, $I_a(X)$ is the maximum ideal defining X.

(ix) If $X \subset X' \subset \mathbb{A}^n$ then $I_a(X') \subset I_a(X)$.

(x) If $\{X_j\}_{j\in J}$ is a collection of subsets of $\mathbb{A}^n$, then $I_a(\bigcup_{j\in J} X_j) = \bigcap_{j\in J} I_a(X_j)$.

(xi) For any $X \subset \mathbb{A}^n$, $X \subset VI_a(X)$, with equality if and only if X is an affine set. In particular $VI_a(X)$ is the minimum affine set containing X.

(xii) For any $T \subset \mathbb{K}[X_0, \dots, X_n]$, $T \subset I_aV(T)$ and $VI_aV(T) = V(T)$.

(xiii) An affine set $X \subset \mathbb{A}^n$ is irreducible if and only if $I_a(X)$ is a prime ideal.

As in the projective case, the main property of the operators V and I_a is the Nullstellensatz. This can be obtained from the projective case (although usually one proves first the affine case and deduce from it the projective theorem).

Theorem 14.2 (Weak Hilbert's Nullstellensatz). *Let $I \subsetneq \mathbb{K}[X_1, \ldots, X_n]$ be a proper ideal. If $\mathbb{K}$ is algebraically closed, then $V(I) \neq \emptyset$.*

Proof: Assume for contradiction that $V(I) = \emptyset$, and let us prove that then I is the whole polynomial ring. If we choose a set of generators $f_1, \ldots, f_r$ of I, then it is clear that our assumption is equivalent to the fact that there is no point $(a_1, \ldots, a_n) \in \mathbb{K}^n$ such that $f_i(a_1, \ldots, a_n) = 0$ for $i = 1, \ldots, r$. Let $F_i \in \mathbb{K}[X_0, \ldots, X_n]$ be the homogenization of each f_i (i.e. $F_i = X_0^{\deg f_i} f_i(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_O})$). It is then obvious that any point $(a_0 : \ldots : a_n) \in V(F_1, \ldots, F_r)$ must satisfy $a_0 = 0$, since otherwise $(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_o})$ would vanish for $f_1, \ldots, f_r$. We thus have that X_0 belongs to $IV(F_1, \ldots, F_r)$. But then Theorem 3.17 implies that there exists a power of X_0 in the ideal $(F_1, \ldots, F_r)$, i.e. we have a relation of the type $X_0^d = G_1F_1 + \ldots + G_rF_r$, with $G_1, \ldots, G_r \in \mathbb{K}[X_0, \ldots, X_n]$. Making $X_0 = 1$ in the above relation we get that $1 = g_1f_1 + \ldots + g_rf_r$, with $g_i = G_i(1, X_1, \ldots, X_n) \in \mathbb{K}[X_1, \ldots, X_n]$. We thus get that 1 belongs to I, and hence $I = \mathbb{K}[X_1, \ldots, X_n]$, which completes the proof. □

Exercise 14.3. Prove directly Theorem 14.2 following the following steps (parallel to those of the proof of Theorem 1.24):

(i) Use induction on n (the case $n = 1$ being trivial from the assumption that $\mathbb{K}$ is algebraically closed and the fact that $k[X_1]$ is a PID).

(ii) If $f \in \mathbb{K}[X_1, \ldots, X_n]$ is a non-constant polynomial, prove that it is possible to find $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{K}$ such that $f(X_1 + \lambda_1 X_n, \ldots, X_{n-1} + \lambda_{n-1}X_n, X_n)$ is monic in the variable X_n (if f_d is the homogeneous component of f of maximum degree, it will be enough to find $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{K}$ such that $f_d(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$).

(iii) If $n > 1$ and $I \subsetneq \mathbb{K}[X_1, \ldots, X_n]$ is a proper ideal, from (ii) we can assume, after changing coordinates that I contains a monic polynomial in X_n, say g; by induction hypothesis, we can find a point $(a_1, \ldots, a_{n-1})$ vanishing at all the polynomials of $I \cap \mathbb{K}[X_1, \ldots, X_{n-1}]$. Prove that $\{f(a_1, \ldots, a_{n-1}, X_n) \mid f \in I\}$ is a proper ideal of $\mathbb{K}[X_n]$ (if there exists $f \in I$ such that $f(a_1, \ldots, a_{n-1}, X_n) = 1$, conclude that the resultant of f and g with respect to X_n cannot vanish at $(a_1, \ldots, a_{n-1})$).

(iv) Conclude from (iii) that there exists $a_n \in \mathbb{K}$ such that $(a_1, \ldots, a_{n-1}, a_n)$ vanishes at all the polynomials of I.

Theorem 14.4 (Hilbert's Nullstellensatz)**.** *Let $I \subset \mathbb{K}[X_1, \ldots, X_n]$ be any ideal. Then $I_aV(I) = \sqrt{I}$.*

Proof: Since clearly $I_aV(I) \supset \sqrt{I}$, we just need to prove the other inclusion. The proof can be obtained easily from the above weak Nullstellensatz by using the trick of Rabinowitsch. Let $f \in \mathbb{K}[X_1, \ldots, X_n]$ be a polynomial in $I_aV(I)$, i.e. vanishing at all the points of $V(I)$. We add a new variable X_{n+1} and consider the polynomial ring $\mathbb{K}[X_1, \ldots, X_{n+1}]$. If $f_1, \ldots, f_r \in \mathbb{K}[X_1, \ldots, X_n]$ is a set of generators of I, it follows from the hypothesis on f that the ideal of $\mathbb{K}[X_1, \ldots, X_{n+1}]$ generated by $f_1, \ldots, f_r$ and $X_{n+1}f - 1$ defines the empty set. Hence Theorem 14.2 implies that there exist $g_1, \ldots, g_{r+1} \in \mathbb{K}[X_1, \ldots, X_{n+1}]$ such that

$$1 = g_1f_1 + \ldots + g_rf_r + g_{r+1}(X_{n+1}f - 1)$$

We now make the substitution $X_{n+1} = \frac{1}{f}$ at each of the polynomials $g_1, \ldots, g_r$. If l is the maximum exponent of f in the denominators of those substitutions, we can thus write $g_i(X_1, \ldots, X_n, \frac{1}{f}) = \frac{h_i}{f^l}$, with $h_i \in \mathbb{K}[X_1, \ldots, X_n]$. Therefore, making the substitution $X_{n+1} = \frac{1}{f}$ in the displayed equation and multiplying by f^l we get the equality $f^l = h_1f_1 + \ldots + h_rf_r$, just proving that f belongs to $\sqrt{I}$, finishing the proof of the theorem. □

The most characteristic result for affine sets (which will be crucial for what follows) was already proved implicitly in Lemma 8.13.

Theorem 14.5. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine sets and let $g \in \mathbb{K}[X_1, \ldots, X_n]$. Then any regular map $X \cap D(g) \to Y$ is defined by m elements of $\mathbb{K}[X_1, \ldots, X_n]_g$. In particular, any regular map $X \to Y$ is defined by m polynomials in $\mathbb{K}[X_1, \ldots, X_n]$.*

Proof: For the last statement, just repeat the proof of Lemma 8.13, identifying $\mathbb{A}^n$ with $D(X_0) \subset \mathbb{P}^n$ and $\mathbb{A}^m$ with $D(Y_0) \subset \mathbb{P}^m$. And now for the general case, we just use the natural isomorphism $D(g) \to V(gX_{n+1} - 1) \subset \mathbb{A}^{n+1}$ defined by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \frac{1}{g(x_1, \ldots, x_n)})$. □

Definition. A *regular function* over a quasiprojective set is a regular map $X \to \mathbb{K}$, where $\mathbb{K}$ is identified with the affine line. The set of regular functions over X will be denoted by $\mathcal{O}(X)$.

As we have seen (Exercise 7.14), $\mathcal{O}(X) = \mathbb{K}$ if X is a projective variety. In the affine case, however, any regular map from $X \subset \mathbb{A}^n$ to $\mathbb{K}$ is defined (after Theorem 14.5) by a polynomial in $\mathbb{K}[X_1, \ldots, X_n]$. Since two polynomials define the same function on X if and only if their difference lies in $I_a(X)$, it follows that $\mathcal{O}(X)$ is naturally isomorphic to $\mathbb{K}[X_1, \ldots, X_n]/I_a(X)$. Moreover, this quotient determines the isomorphism class of the affine set, as the following result shows.

Theorem 14.6. *If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine sets, then there is a natural bijection between the set of regular morphisms from X to Y and the set of homomorphisms of $\mathbb{K}$-algebras from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$. Moreover, this bijection defines a 1:1 correspondence between isomorphisms of the affine sets and isomorphisms of the $\mathbb{K}$-algebras. Hence there is a natural bijection between the set of isomorphism classes of affine sets and the set of isomorphism classes of finitely generated reduced $\mathbb{K}$-algebras.*

Proof: If $f : X \to Y$ is a regular morphism, we know that it is defined by the classes (modulo $I_a(X)$) of m polynomial $f_1, \ldots, f_m \in \mathbb{K}[X_1, \ldots, X_n]$. We can thus define a homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ by assigning to the class of Y_i (modulo $I_a(Y)$) the class of f_i (modulo $I_a(X)$). Viceversa, given a homomorphism of $\mathbb{K}$-algebras $\varphi : \mathcal{O}(Y) \to \mathcal{O}(X)$, we can take $f_1, \ldots, f_m \in \mathbb{K}[X_1, \ldots, X_n]$ representatives modulo $I_a(X)$ of the respective images by φ of the classes (modulo $I_a(Y)$) of $Y_1, \ldots, Y_m$. It is clear that this defines the wanted bijection. Since for morphisms defined on the appropriate affine sets it holds $(f \circ g)^* = g^* \circ f^*$ and $(id_X)^* = id_{\mathcal{O}(X)}$, then it follows immediately that this bijection takes isomorphisms to isomorphisms.

For the last statement of the theorem, first observe that for any affine set $X \subset \mathbb{A}^n$, the $\mathbb{K}$-algebra $\mathcal{O}(X)$ is finitely generated (by the classes of $X_1, \ldots, X_n$). Reciprocally, if A is a finitely generated $\mathbb{K}$-algebra, let $\alpha_1, \ldots, \alpha_n$ be a set of generators. We therefore have a surjective homomorphism $\mathbb{K}[X_1, \ldots, X_n] \to A$ that associates to each X_i the generatorα_i. If I is the kernel of that map, then I is a radical ideal, because A is reduced. But then the Hilbert's Nullstellensatz implies that $I = I_a V(I)$, so that A is isomorphic to $\mathcal{O}(V(I))$. This completes the proof of the theorem. □

Remark 14.7. From Theorem 14.5, it also follows that for any basic open set $X \cap D(g)$ (for short we will denote it $D_X(g)$), its set of regular functions is naturally isomorphic to $\mathcal{O}(X)_g$, the set of quotients $\frac{f}{g^l}$, with $f \in \mathcal{O}(X)$ and $l \in \mathbb{N}$.

Exercise 14.8. Show that, if $X = \mathbb{A}^2 \setminus \{(0,0)\}$, then $\mathcal{O}(X)$ is naturally isomorphic to $\mathbb{K}[X_1, X_2]$. [Hint: Given a regular function on X, restrict it to $D(X_1)$ and $D(X_2)$, use the above remark and compare both restrictions].

We would like to have something like Theorem 14.6 for projective sets, i.e. find some algebraic object characterizing a projective set up to isomorphism. This object cannot be the set of regular functions, since the only regular functions of any projective variety are the constant maps. This object cannot be neither the graded ring, since isomorphic varieties can have non isomorphic graded rings (consider for instance two rational normal curves of different degrees; they are isomorphic since both are isomorphic to $\mathbb{P}^1$, but their Hilbert polynomials are different, so that their graded rings cannot be isomorphic). It seems therefore that the only way of solving this problem is to consider all the affine pieces of a projective set in order to determine its isomorphism class. This is the motivation for the following definition (which we will give with some generality).

Definition. A *sheaf of* $\mathbb{K}$*-algebras over a topological space* X is a map $\mathcal{O}$ from the set of open sets of X to the set of $\mathbb{K}$-algebras satisfying the following conditions:

(i) $\mathcal{O}(\emptyset) = 0$.

(ii) If $V \subset U$ are two open sets of X, there is a homomorphism of $\mathbb{K}$-algebras (which we will call *restriction map*) $\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$ (we will often write $\rho_{UV}(f) = f_{|V}$).

(iii) For any open set $U \subset X$, ρ_{UU} is the identity map.

(iv) If $W \subset V \subset U$, then $\rho_{UW} = \rho_{V_W} \circ \rho_{UV}$.

(v) If an open set U is the union of open sets U_i (with i varying in an arbitrary set I), and for each $i \in I$ there is $f_i \in \mathcal{O}(U_i)$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique $f \in \mathcal{O}(U)$ such that $f_{|U_i} = f_i$ for any $i \in I$.

The elements of $\mathcal{O}(U)$ are called *sections of the sheaf* $\mathcal{O}$ over U. Clearly, this definition can be extended to sheaves of rings, groups,... by just modifying conveniently the definition. In particular, a *sheaf of* $\mathcal{O}$*-modules* will consist of a sheaf $\mathcal{F}$ such that for each open set $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-module, and for each $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a homomorphism of $\mathcal{O}(U)$-modules (note that the restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ for $\mathcal{O}$ endows $\mathcal{F}(V)$ with a structure of $\mathcal{O}(U)$-module).

Exercise 14.9. Prove that, if an open set U of X is covered by a collection $\{U_i\}_{i \in I}$ of open sets and there are $f, g \in \mathcal{O}(U)$ such that for each $i \in I$ it holds $f_{|U_i} = g_{|U_i}$, then $f = g$. [Hint: Use the fact that $0 \in \mathcal{O}(U)$ is the only section whose restriction to each U_i is zero].

Definition. If X is a quasiprojective set, the sheaf $\mathcal{O}_X$ that associates to each open set $U \subset X$ the set of regular functions over U is a sheaf over X, called the *structure sheaf of the quasiprojective set.*

The idea now is that two quasiprojective sets are isomorphic if and only if they have isomorphic –in some sense– structure sheaves. This is what we want to do now. We first observe, that if we have a regular map $f : X \to X'$ among two quasiprojective sets, then for any open set $U' \subset X'$ we have a homomorphism of $\mathbb{K}$-algebras $\mathcal{O}_{X'}(U') \to \mathcal{O}_X(f^{-1}(U'))$. This motivates the following definitions (in which you can take your favorite algebraic structure for the sheaves).

Definition. Let X, Y be two topological spaces, let $f : X \to Y$ be a continuous map and let $\mathcal{F}$ be a sheaf over X. Then the *direct image* sheaf $f_*\mathcal{F}$ is the sheaf over Y defined by

$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ (it is an easy exercise left to the reader that this is a sheaf with the obvious restriction maps).

Definition. Let X be a topological space and let $\mathcal{F}$, $\mathcal{F}'$ be two sheaves over X. A *morphism of sheaves* $\varphi : \mathcal{F} \to \mathcal{F}'$ is a collection of homomorphisms (with respect to the structure you chose) $\varphi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ one for each open set $U \subset X$ and such that they are compatible with the restrictions: if $V \subset U$ then $\rho'_{UV} \circ \varphi_U = \varphi_V \circ \rho_{UV}$. The composition of two morphisms is defined in the obvious way. An *isomorphism of sheaves* is a morphism $\varphi : \mathcal{F} \to \mathcal{F}'$ such that there exists another morphisms $\psi : \mathcal{F}' \to \mathcal{F}$ satisfying $\varphi \circ \psi = id_{\mathcal{F}'}$ and $\psi \circ \varphi = id_{\mathcal{F}}$.

We can state and prove now a generalization of Theorem 14.6.

Theorem 14.10. *Let X and Y two quasiprojective varieties and let $f : X \to Y$ be a regular map. Then f is an isomorphism if and only if the inverse image by f of any affine open set of Y is affine and the natural morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

Proof: The "only if" part is obvious, so we will only proof the "if" part. We cover Y by a collection $\{V_i\}$ of affine open sets. For each i we write $U_i = f^{-1}(V_i)$, which is affine by hypothesis. Also by hypothesis we have that $f^* : \mathcal{O}(V_i) \to \mathcal{O}(U_i)$ is an isomorphism. Therefore Theorem 14.6 implies that $f_{|U_i} : U_i \to V_i$ is an isomorphism. This immediately implies that f is an isomorphism, as wanted. □

Exercise 14.11. Show that, if $X = \mathbb{A}^2 \setminus \{(0,0)\}$ and $f : X \to \mathbb{A}^2$ is the inclusion, then the morphism $\mathcal{O}_{\mathbb{A}^2} \to f_*\mathcal{O}_X$ is an isomorphism. Therefore the hypothesis about the inverse image of the affine sets is necessary in the above theorem.

We finish this chapter with a technical result about sheaves that will be extremely useful in the next chapters.

Proposition 14.12. *Let X be a topological space with basis B. Assume that for each $U \in B$ we have a group $\mathcal{F}(U)$ (or ring, or any other algebraic structure). Assume also that if we have $V \subset U$ with $V, U \in B$ then we have a homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ and that for $W \subset V \subset U$ it holds $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. Then there exists a unique (up to isomorphism) sheaf $\mathcal{F}$ over X whose groups and restriction maps coincide with the given ones for the open sets in B.*

Proof: For any open set $U \subset X$, define $\mathcal{F}(U)$ as the subset of the product $\Pi_{V \in B, V \subset U} \mathcal{F}(V)$ consisting of elements $(\{s_V\})$ for which $\rho_{VW}(s_V) = s_W$ if $W \subset V \subset U$ and $W, V \in B$. It is an easy but tedious exercise that $\mathcal{F}$ is a sheaf with the natural restriction maps, and that it satisfies the conditions of the proposition. □

15. The local ring at a point

The idea of this chapter is to work with germs of functions around a point, in the same sense as in analytic geometry. In fact we will see that it is possible to recover with an algebraic language the notion of Taylor series.

Definition. Let X be a quasiprojective set and let x be a point of it. Consider the set of pairs (U, f) such that U is an open set of X containing x and f is a regular function on U. We say that two pairs (U', f') and (U'', f'') are equivalent if there exists another pair (U, f) such that $U \subset U' \cap U''$ and $f'_{|U} = f''_{|U} = s$. This is an equivalence relation and an equivalence class is called a *germ of regular function at the point* x. The set of germs of regular functions of X is called the *local ring of* X *at* x, and will be denoted by $O_{X,x}$.

Proposition 15.1. *Let X be a quasiprojective set and let x be a point of it.*

(i) If $U \subset$ is an open set containing x, then $\mathcal{O}_{X,x} = \mathcal{O}_{U,x}$.

(ii) If X is an affine set with affine coordinate ring $\mathcal{O}(X)$ and I_x is the ideal of all the regular functions vanishing at x, then $\mathcal{O}_{X,x}$ is naturally isomorphic to the localization $(\mathcal{O}(X))_{I_x}$.

(iii) If X is a projective set with coordinate ring $S(X)$ and I_x is the class modulo $I(X)$ of the ideal $I(x)$, then $\mathcal{O}_{X,x}$ is naturally isomorphic to the localization $(S(X))_{(I_x)}$.

(iv) The ring $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{M}_x$, the set of germs vanishing at x.

Proof: Part (i) is obvious from the definition of germ, while part (iv) is an immediate consequence of (ii). Since parts (ii) and (iii) are proved in a similar way, we will only prove part (iii).

Clearly an element $\frac{F}{G} \in S(X)_{(I_x)}$ defines the germ represented by the pair $(D(G), \frac{F}{G})$. Reciprocally, if (U, f) represents a germ of regular function at x, then there exists homogeneous polynomials F, G of the same degree such that f is represented near x by $\frac{F}{G}$. It is very easy to check that these two operations define maps between $(S(X))_{(I_x)}$ and $\mathcal{O}_{X,x}$ that are inverse to each other. $\square$

Definition. A *germ of algebraic subset of a quasiprojective set* X *at a point* x is an equivalence class of quasiprojective set $Y \subset X$ containing x, in which two subsets $Y, Y' \subset X$ are said to be equivalent if and only if they coincide when restricted to some open set of X containing x.

It is clear that we can extend to germs most of the definitions and results that we have for quasiprojective sets.

Exercise 15.2. If $Y \subset X$ is a germ of algebraic set of X, define its ideal $I_x(Y) \subset \mathcal{O}_{X,x}$ as the set of germs of functions vanishing at Y. Reciprocally, for any set T of germs of functions let $V_x(T)$ be the germ of algebraic set of X defined by the vanishing of the elements of X.

(i) Prove that I_x and V_x satisfy the properties of Proposition 1.1.

(ii) Prove that a germ Y of algebraic subset is irreducible (in the obvious sense) if and only if $I_x(Y)$ is prime. Show that any germ of algebraic subsets of X can be decomposed in a unique non-trivial way as a union of irreducible germs.

(iii) Prove the local Nullstellensatz: $I_xV_x(I) = \sqrt{I}$.

(iv) We define the dimension of a germ of algebraic sets X as the one of any projective set defining it and such that all of its components pass through x. Prove that the dimension of X is the maximum length of a chain of prime ideals of $\mathcal{O}_{X,x}$ (you will need to adapt the proof of Proposition 5.7 by using Theorem 7.21).

With this point of view, observe that the proof of Theorem 11.1 shows that the maximal ideal $\mathfrak{M}_x$ of $\mathcal{O}_{X,x}$ is generated by the classes modulo $I(X)$ of r independent linear forms generating a linear space meeting $\mathbb{T}_xX$ (of dimension r) exactly at the point x. It is interesting to observe that, instead of working modulo the whole ideal $I(X)$, we just used $n-r$ polynomials in $I(X)$ that define the tangent space. A posteriori, it will be essentially the same (as germs), but in order to prove it we will need to keep for a while this distinction.

Definition. Let X be a germ of algebraic subset of $\mathbb{P}^n$ at a point x, and assume that X is smooth of local dimension r at x. A *smooth system of equations of* X at x is a set of elements $f_{r+1},\dots,f_n \in I_x(X) \subset \mathcal{O}_{\mathbb{P}^n,x}$ such that their classes in $\mathfrak{M}_x/\mathfrak{M}_x^2$ are linearly independent (in other words, they generate $\mathbb{T}_xX$). A *local system of parameters for the smooth equations* $f_{r+1},\dots,f_n$ at x is a set of r generators of the maximal ideal of $\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1},\dots,f_n)$.

Theorem 15.3. *Let X be a germ of algebraic set in $\mathbb{P}^n$, let $x \in X$ be a smooth point with smooth equations $f_{r+1},\dots,f_n$, and let $u_1,\dots,u_r$ a local system of parameters for them at x. Then for any $f \in \mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1},\dots,f_n)$ there exists a unique formal series $\sum a_{i_1\dots i_r}T_1^{i_1}\dots T_r^{i_r}$ such that for each $s \in \mathbb{N}$ it holds $f - \sum_{i_1+\dots+i_r\le s} a_{i_1\dots i_r}u_1^{i_1}\dots u_r^{i_r} \in \mathfrak{M}_x^{s+1}$.*

Proof: We prove first the existence. Given $f \in \mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1},\dots,f_n)$, we define $a_{0\dots0} = f(x)$. It thus follows that $f - a_{0\dots0}$ belongs to $\mathfrak{M}_x$. It is thus possible to write $f - a_{0\dots0} = f_1u_1 + \dots + f_ru_r$, for some $f_1,\dots,f_r \in \mathcal{O}_{X,x}$. We write now $a_{10\dots0} = f_1(x)$, $\dots$, $a_{0\dots01} = f_r(x)$, and it follows that $f - a_{0\dots0} - a_{10\dots0}u_1 - \dots - a_{0\dots01}u_r = (f_1 - a_{10\dots0})u_1 +$

$\ldots + (f_r - a_{0\ldots01})u_r$ is in $\mathfrak{M}_x^2$. Therefore it can be written as $\sum_{i_1+\ldots+i_r=2} f_{i_1\ldots i_r} u_1^{i_1}\ldots u_r^{i_r}$. Iterating this construction, we prove the existence of the formal series.

To prove the uniqueness, it is enough to prove it for the zero function. In other words, we have to prove that if $\sum_{i_1+\ldots+i_r\leq s} a_{i_1\ldots i_r} u_1^{i_1}\ldots u_r^{i_r} \in \mathfrak{M}_x^{s+1}$ for each s, then all the coefficients $a_{i_1\ldots i_r}$ are zero. This is equivalent to prove that if a homogeneous polynomial $P\in\mathbb{K}[T_1,\ldots,T_r]$ verifies that $P(u_1,\ldots,u_r)$ is in $\mathfrak{M}_x^x$ then P must be zero. Assume for contradiction that we have such a nonzero polynomial P. If necessary, we change u_r by a general linear combination of $u_1,\ldots,u_r$, and we can assume that P is monic in T_r. We thus have a relation $u_r^s - f_1u_r^{s-1} - \ldots - f_s \in \mathfrak{M}_x^{s+1}$, where each f_i is a homogeneous polynomial of degree i in $u_1,\ldots,u_{r-1}$. On the other hand, an element of $\mathfrak{M}_x^{s+1}$ can be written as $u_r^s f + g$, where f is in $\mathfrak{M}_x$ and g is in the ideal generated by $(u_1,\ldots,u_{r-1}$. We have then that $u_r^s(1-f)$ is in the ideal generated by $u_1,\ldots,u_{r-1}$, and since $1-f\notin\mathfrak{M}_x$, the same holds for u_r^s. But this means that the local functions $u_1,\ldots,u_r$ vanish near x exactly where $u_1,\ldots,u_{r-1}$ do. This is impossible since $u_1,\ldots,u_{r-1}$ cannot vanish only at x. □

Definition. The formal series that the previous theorem assigns to a local function f near $x\in X$ is called *the Taylor series of the function f at the point x* with respect to $f_{r+1},\ldots,f_n$ and $u_1,\ldots,u_r$.

Proposition 15.4. *If the Taylor series of a local function f at a smooth point x of a germ of algebraic set X is zero, then the function itself is zero.*

Proof: Let $I\subset\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1},\ldots,f_n)$ be the set of local functions whose Taylor series is zero. This is clearly an ideal of $\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1},\ldots,f_n)$, and in fact it is $I=\cap_{s\geq0}\mathfrak{M}_x^s$. We will prove by induction on the local dimension r of X at x that I is the zero ideal. If $r=0$ there is nothing to prove.

Assume now that $r>0$. We will consider X' to be the germ of algebraic set defined as $X\cap V(u_r)$. By Theorem 7.21, X' has local dimension $r-1$ at x. Since $\mathbb{T}_xX'$ is contained in $T_xX\cap\mathbb{T}_xV(u_r)$, which has dimension $r-1$, it follows that X' is smooth at x and $u_1,\ldots,u_{r-1}$ is a local system of parameters of X' at x. Hence, for any $f\in I$, since its class modulo (u_r) has also zero Taylor series, by induction hypothesis it follows that $f=gu_r$ for some $g\in\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1},\ldots,f_n)$, and clearly g must also have a zero Taylor series. The idea is to apply this to a set of generators of I.

Since $\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1},\ldots,f_n)$ is noetherian, the ideal I has a finite set of generators, say $k_1,\ldots,k_s$. As we have remarked before, we can write $k_i=g_iu_r$, for some $g_i\in I$. We can thus write $g_i = h_{i1}k_1 + \ldots + h_{is}k_s$. There is therefore a matrix relation

$$\begin{pmatrix} 1-h_{11}u_r & -h_{12}u_r & \ldots & -h_{1s}u_r \\ -h_{21}u_r & 1-h_{22}u_r & \ldots & -h_{2s}u_r \\ \vdots & \vdots & \ddots & \vdots \\ -h_{s1}u_r & -h_{s2}u_r & \ldots & 1-h_{ss}u_r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

If h is the determinant of the left-hand matrix, the multiplication of the above expression by the adjoint of that matrix yields $hk_i = 0$ for $i = 1, \ldots, s$. Since $h(x) = 1$, it follows that h is a unit in $\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1}, \ldots, f_n)$, and therefore $k_1 = \ldots = k_s = 0$, which proves $I = 0$. □

Corollary 15.5. *Let X be a germ of algebraic set that is smooth at x, and let $f_{r+1}, \ldots, f_n$ be a smooth system of equations. Then*

(i) The ideal $(f_{r+1}, \ldots, f_n) \subset \mathcal{O}_{\mathbb{P}^n,x}$ is prime.

(ii) The germ $V(f_{r+1}, \ldots, f_n)$ coincides with X, and it has only one component.

(iii) The natural projection $\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1}, \ldots, f_n) \to \mathcal{O}_{X,x}$ is an isomorphism.

Proof: By Theorem 15.3 and Proposition 15.4, $\mathcal{O}_{\mathbb{P}^n,x}/(f_{r+1}, \ldots, f_n)$ is isomorphic to a subring of $\mathbb{K}[[T_1, \ldots, T_r]]$, which is an integral domain. Therefore, $(f_{r+1}, \ldots, f_n)$ is a prime ideal, proving (i). Since $V(f_{r+1}, \ldots, f_n)$ is therefore irreducible of dimension r and contains X, which has also dimension r, they have to coincide, which proves (ii). Hence $I_x(X)$ is the radical of $(f_{r+1}, \ldots, f_n)$, but the latter being prime it coincides with its radical. This proves (iii). □

Theorem 15.6. *Let $X \subset \mathbb{P}^n$ be a projective set and let $x \in X$ be a smooth point. Consider the map $\nu_x : \mathcal{O}_{X,x} \setminus \{0\} \to \mathbb{Z}$ that assigns to each $f \in \mathcal{O}_{X,x} \setminus \{0\}$ the only integer $\nu_x(f)$ such that $f \in \mathfrak{M}_x^{\nu_x(f)} \setminus \mathfrak{M}_x^{\nu_x(f)+1}$. Then*

(i) $\nu_x(fg) = \nu_x(f) + \nu_x(g)$.

(ii) $\nu_x(f+g) \geq \min\{\nu_x(f), \nu_x(g)\}$, with equality only if $\nu_x(f) \neq \nu_x(g)$.

(iii) $\nu_x(f) > 0$, if and only if $f \in \mathfrak{M}_x$.

Proof: Since $\nu_x(f)$ can be defined as the order of a Taylor series of f with respect to any local system of parameters, the proof is obvious. □

16. Introduction to affine and projective schemes

In this chapter, using the philosophy of the previous results, we will give a first approach to the theory of schemes. The definition we will give here is not the correct one, but it will be enough to study schemes with support a quasiprojective set.

Example 16.1. We start by trying to give a precise structure in situations like Example 1.21, in which we have an ideal representing not only a point, but a tangent direction. Putting this example in the affine plane we can consider the intersection of the parabola $V(X_2 - X_1^2)$ and its tangent line $V(X_2)$. We thus get the point $(0,0)$, but defined by the ideal (X_1^2, X_2). Since by Theorem 14.6 a radical ideal is the same as an affine set, and the set of regular functions is the quotient of the polynomial ring by this ideal, we can now define a new object that, as a set, consists of a single point, say $(0,0)$, but whose set of regular functions is $\mathbb{K}[X_1,X_2]/(X_1^2,X_2)$. In this way, a morphism from this object to $\mathbb{A}^n$ should consist of a homomorphism $\varphi : \mathbb{K}[Y_1,\dots,Y_n] \to \mathbb{K}[X_1,X_2]/(X_1^2,X_2)$. This homomorphism will be determined by the image of each Y_i, which will be the class of some $a_i + b_iX_1$ (with $a_i, b_i \in \mathbb{K}$). This can be regarded as a sort of Taylor expansion up to degree one, indicating that the image of this new object consists not only of the image of the point $(0,0)$ –which is the point $(a_1,\dots,a_n)$– but also of the vector $(b_1,\dots,b_n)$, corresponding with the intuitive idea that our object represents not only a point, but also a tangent direction.

In the case of the concrete Example 1.21, the new structure on the point $(1:0:0)$ must be given, in the flavor of Theorem 14.10 by defining a sheaf $\mathcal{O}$ that on each open set gives the set of what we want to be now the regular functions. For instance, for $D(X_0)$ we have to put the ring in the above case, which is nothing but the subring of $\big(\mathbb{K}[X_0,X_1,X_2]/I\big)_{X_0}$ (where $I = (X_1^2, X_2)$) of quotients of a homogeneous element of degree say d by X_0^d. In a similar way, in the open set $D(F)$, with F homogeneous of degree e we define $\mathcal{O}(D(F)$ as the subring of $\big(\mathbb{K}[X_0,X_1,X_2]/I\big)_F$ of quotients of a homogeneous polynomial of some degree de by F^d. And for any open set U, $\mathcal{O}(U)$ can be defined as $\mathcal{O}(D(F))$ for any $D(F)$ containing $(1:0:0)$ and contained in U.

Definition. A *scheme structure over a quasiprojective set* X_{red} will be a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$ over X_{red} such that for each open set $U \subset X_{red}$ the quotient of $\mathcal{O}_X(U)$ by the nilradical is $\mathcal{O}_{X_{red}}(U)$. We will usually denote by X a scheme. The quasiprojective set X_{red} will be called the *(reduced) support of the scheme.*

Definition. Let $I \subset \mathbb{K}[X_1,\dots,X_n]$ an ideal and write $X_{red} = V(I)$. For each $f \in \mathbb{K}[X_1,\dots,X_n]$, we define $\mathcal{O}_X(D_{X_{red}}(f)) = \big(\mathbb{K}[X_1,\dots,X_n]/I\big)_f$. Then this collection of $\mathbb{K}$-algebras satisfies the hypotheses of Proposition 14.12, so that it defines a sheaf of $\mathbb{K}$-

algebras $\mathcal{O}_X$ over X_{red}. The sheaf $\mathcal{O}_X$ induces a scheme structure over X_{red}, which we will call the *scheme induced by* I.

Remark 16.2. Notice that, in the above definition, the projection $\mathbb{K}[X_1,\dots,X_n] \to \mathbb{K}[X_1,\dots,X_n]/I$ induces epimorphisms in its localizations by polynomials. We thus have a morphism of sheaves $\varphi : \mathcal{O}_{\mathbb{A}^n} \to i_*\mathcal{O}_X$ (where $i : X_{red} \hookrightarrow \mathbb{A}^n$ is the inclusion map) such that the maps $\varphi_{D(f)}$ are all surjective. Observe that this latter condition does not imply that all the maps φ_U are surjective. For instance, if $X = X_{red} = V(X_2) \subset \mathbb{A}^2$ and $U = \mathbb{A}^2 \setminus \{(0,0)\}$, the map φ_U is the natural map $\mathbb{K}[X_1, X_2] \to \mathbb{K}[X_1]_{X_1}$ (see Exercise 14.11), which is not surjective.

Definition. An *embedded scheme structure over an affine set* $X_{red} \subset \mathbb{A}^n$ is a scheme structure X such that there is a morphism of sheaves $\varphi : \mathcal{O}_{\mathbb{A}^n} \to i_*\mathcal{O}_X$ satisfying that, for any $f \in \mathbb{K}[X_1,\dots,X_n]$, the map $\varphi_{D(f)}$ is surjective.

It is clear that any embedded scheme structure over X_{red} comes from an ideal I (precisely the kernel of $\varphi_{\mathbb{A}^n}$) and that two ideals define the same embedded scheme structure if and only they are equal.

Definition. Let $I \subset \mathbb{K}[X_0,\dots,X_n]$ a homogeneous ideal and let $X_{red} = V(I)$. For each homogeneous polynomial $F \in \mathbb{K}[X_0,\dots,X_n]$, define $\mathcal{O}_X(D_{X_{red}}(F)) = \big(\mathbb{K}[X_0,\dots,X_n]/I\big)_{(F)}$. Again this collection of $\mathbb{K}$-algebras satisfies the hypotheses of Proposition 14.12, so that it defines a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$ over X_{red}. The sheaf $\mathcal{O}_X$ induces a scheme structure over X_{red}, which we will call the *embedded scheme induced by* I.

Remark 16.3. Notice that the above definition implies that the restriction of the scheme X to the affine set $D(X_i)$ is precisely the scheme constructed for the affine set defined by the dehomogenization of I with respect to X_i.

Proposition 16.4. *Any embedded scheme structure over a projective set $X_{red} \subset \mathbb{P}^n$ is induced by a homogeneous ideal $I \subset \mathbb{K}[X_0,\dots,X_n]$. Moreover, the embedded scheme structures defined by two ideals I and I' are the same if and only if I and I' have the same saturation, i.e. if there exists l_0 such that $I_l = I'_l$ for $l \geq l_0$.*

Proof: Let X be an embedded scheme structure over X_{red}. We define $I \subset \mathbb{K}[X_0,\dots,X_n]$ to be the ideal generated by all the homogeneous polynomials G (of degree say d) such for each $i = 0,\dots,n$, the quotient $\frac{F}{X_i^d}$ is in the kernel of $\varphi_{D(X_i)}$. It is not difficult to see that I induces the embedded scheme structure X.

On the other hand, if two homogeneous ideals $, I'$ induce the same scheme structure on X_{red}, then from Remark 16.3 it follows that the dehomogenizations of I and I' with respect to X_i are the same for each $i = 0,\dots,n$. This is equivalent to say that I and I' have the same saturation, which completes the proof by using Proposition 2.12. $\square$

We come now back to Remark 16.2, specifically to the fact that we can have a morphism of sheaves that is surjective when evaluated at a basis of the topology but it is not surjective when evaluated at all the open sets. This seems to be in contradiction with Proposition 14.12 (at least heuristically). The reason is that the image of a morphism of sheaves cannot be defined in its natural way. We will understand this better by comparing it with the good behavior of kernels.

Exercise 16.5. Let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a morphism of sheaves. Prove that the assignment $U \mapsto \ker(\varphi_U)$ defines a sheaf over X.

Definition. The above sheaf is called the *kernel of the morphism* φ, and it is denoted by $\mathcal{K}er\varphi$. The morphism φ is said to be an *injective morphism of sheaves* if the sheaf $\mathcal{K}er\varphi$ is the zero sheaf. By the previous exercise, this is equivalent to say that all the maps φ_U are injective.

Example 16.6. Consider again the example of Remark 16.2, in which we had the inclusion $i : X = V(X_2) \hookrightarrow \mathbb{A}^2$ and the morphism $\varphi : \mathcal{O}^2_{\mathbb{A}} \to i_*\mathcal{O}_X$ of sheaves over $\mathbb{A}^2$. Consider the open set $U = \mathbb{A}^2 \setminus \{(0,0)\}$. We can cover U by the open sets $U_1 = D(X_1)$ and $U_2 = D(X_2)$. We choose the elements $\frac{1}{X_1} \in (i_*\mathcal{O}_X)(U_1)$ and $0 \in (i_*\mathcal{O}_X)(U_2)$ (observe that this last set is fact zero). They are clearly in the respective images of φ_{U_1} and φ_{U_2}. On the other hand they both restrict to zero in $(i_*\mathcal{O}_X)(U_1 \cap U_2)$. The only element in $(i_*\mathcal{O}_X)(U)$ that restricts to $\frac{1}{X_1}$ and 0 is $\frac{1}{X_1}$. But the latter is not in the image of $\varphi_U : \mathbb{K}[X_1, X_2] \to \mathbb{K}[X_1]_{X_1}$. Therefore the assignment $U \mapsto \mathrm{Im}(\varphi_U)$ does not satisfy the conditions to be a sheaf.

The way of solving our problem is quite natural. As we have observed (Proposition 14.12), a sheaf is determined by a basis of the topology, in other words by sufficiently "small" open sets. However, this is not a natural notion (and the above example shows that with a basis we cannot define the sheaf image). What is natural is to work with sufficiently small open sets near a point, and this is what we are going to do. The example to have in mind is the one of analytic functions, whose behavior in a sufficiently small neighborhood at a point is given by its Taylor expansion, which is a germ of function. This is what we are going to generalize now.

Definition. Let X be a topological space, fix x to be a point of it and let $\mathcal{F}$ be a sheaf over X. Consider the set of pairs (U, s) such that U is an open set of X containing x and $s \in \mathcal{F}(U)$. We say that two pairs (U', s') and (U'', s'') are equivalent if there exists another pair (U, s) such that $U \subset U' \cap U''$ and $s'_{|U} = s''_{|U} = s$. This is an equivalence relation and an equivalence class is called a *germ of section of the sheaf* $\mathcal{F}$ *at the point* x. The set of germs of sections of $\mathcal{F}$ at the point x is called the *stalk of the sheaf* $\mathcal{F}$ *at the point* x and it is denoted by $\mathcal{F}_x$.

Proposition 16.7. *Let $X_{red} \subset \mathbb{A}^n$ be an affine set and let $\mathcal{O}_X$ be the scheme structure induced by an ideal $I \subset R = \mathbb{K}[X_1, \ldots, X_n]$. If x is a point of X and $I_x \subset R/I$ is the maximal ideal corresponding to x, then the stalk of $\mathcal{O}_X$ at x is naturally isomorphic to the localization $(R/I)_{I_x}$.*

Proof: An element of $(R/I)_{I_x}$ is the quotient $\frac{g}{f}$ of (the classes modulo I of) a polynomial g and a polynomial $f \notin I_x$, i.e. $f(x) \neq 0$. We have thus the germ defined by the pair $(D_X(f), \frac{g}{f})$. This defines a natural map from $(R/I)_{I_x}$ to the stalk of $\mathcal{O}_X$ at x, and it is easy to see that it is a homomorphism. We have to construct now an inverse to this map.

Let (U, s) be a representative of a germ of $\mathcal{O}_X$ at x. We can certainly take U to be of the type $U = D_X(f)$ with f a regular function on X not vanishing at x (i.e. $f \notin I_x$). But since $\mathcal{O}_X(D_X(f))$ is by definition $(R/I)_f$, then s can be written as $\frac{g}{f^m}$, which represents in particular an element of $(R/I)_{I_x}$. On the other hand, if $(D_X(f), \frac{g}{f^m})$ and $(D_X(f'), \frac{g'}{f'^m})$ represent the same germ (clearly we can take the same exponent m in the denominators), then there exists some $D_X(h) \subset D_X(f) \cap D_X(f')$ containing x on which $\frac{g}{f^m}$ and $\frac{g'}{f'^m}$ coincide as sections. The inclusion $D_X(h) \subset D_X(ff')$ is equivalent to the inclusion $V(I+(ff')) \subset V(h)$, and therefore by the affine Nullstellensatz we find a relation $h^s = kff'$ for some $s \in \mathbb{N}$ and $k \in R/I$. Hence, as elements of $\mathcal{O}_X(D_X(h)) = (R/I)_h$ we have $(\frac{g}{f^m})_{|D_X(h)} = \frac{k^m f'^m g}{h^m}$ and $(\frac{g'}{f'^m})_{|D_X(h)} = \frac{k^m f^m g'}{h^m}$. Since they coincide, we have the following sequence of equalities in $(R/I)_{I_x}$:

$$\frac{g}{f^m} = \frac{k^m f'^m g}{h^m} = \frac{k^m f^m g'}{h^m} = \frac{g'}{f'^m}$$

This shows that we have a well-defined map from the stalk of $\mathcal{O}_X$ at x and $(R/I)_{I_x}$, and it is very easy to check that it is the inverse of the above map. □

Remark 16.8. The reader is probably wondering what Taylor expansions has to do with all this. Consider on $\mathbb{A}^n$ the sheaf $\mathcal{O}_{\mathbb{A}^n}$ of regular functions (which is the sheaf structure corresponding to the zero ideal). If $x = (0, \ldots, 0)$, then $\mathcal{O}_{\mathbb{A}^n,x}$ is the set of quotients $\frac{f}{g}$, with f and g polynomials such that $g(0, \ldots, 0) \neq 0$. It is then not difficult to check that $\mathcal{O}_x$ is naturally contained in the set $\mathbb{K}[[X_1, \ldots, X_n]]$ of formal (infinite) series $\sum a_{i_1 \ldots i_n} X_1^{i_1} \ldots X_n^{i_n}$. If $\mathbb{K} = \mathbb{C}$ then the inclusion is given by the Taylor expansion of $\frac{f}{g}$ at the origin. In general, the inclusion is given by multiplying f by the formal inverse of g (it can be seen that an element of $\mathbb{K}[X_1, \ldots, X_n]$ is invertible if and only if it does not vanish at the origin). In general, if x is a smooth point of an affine set X of dimension r, it can be seen that $\mathcal{O}_{X,x}$ is also contained inside some ring $\mathbb{K}[Y_1, \ldots, Y_r]$ (see [Sh]). This time the inclusion is not natural, but it is obtained by taking the series expansion with respect to some system of r parameters.

Proposition 16.9. *Let $X_{red} \subset \mathbb{P}^n$ be a projective variety and let $\mathcal{O}_X$ be the scheme structure induced by the homogeneous ideal $I \subset S = \mathbb{K}[X_0, \dots, X_n]$. If x is a point of X_{red} and $I_x \subset S/I$ is the prime ideal corresponding to x, then the stalk of $\mathcal{O}_X$ at x is naturally isomorphic to the localization $(S/I)_{(I_x)}$ consisting of quotients $\frac{G}{F}$, where F and G are (classes modulo I of) homogeneous of the same degree and $F \notin I$.*

Proof: It is straightforward, by just copying the proof of Proposition 16.7 or by restricting $\mathcal{O}_X$ to some $D(X_i)$ containing x and using that this gives the scheme structure defined by $I_{(X_i)}$, i.e. the dehomogenization of I with respect to X_i (see Remark 16.3). □

The idea now is to create sheaves just from the stalks, in the same way as we created them from a basis in (the proof of) Proposition 14.12. This analogy comes from the following result.

Proposition 16.10. *Let $\mathcal{F}$ be a sheaf over a topological space X. Then for any open set $U \subset X$ there is a natural isomorphism between the $\mathcal{F}(U)$ and the subset $\mathcal{F}^+(U)$ of $\Pi_{x \in U} \mathcal{F}_x$ consisting of uples $(s_x)_{x \in U}$ such that for any $x \in U$ there exists an open set $V \subset U$ containing x and an element $s \in \mathcal{F}(V)$ with the property that, for any $y \in V$, the germ s_y is the equivalence class of (V, s).*

Proof: It is clear that we have a homomorphism associating to each $s \in \mathcal{F}(U)$ the uple $(s_x)_{x \in U}$ in which for each $x \in U$ the germ s_x is given by the equivalence class of (U, s). The fact that this homomorphism is bijective is precisely property (v) in the definition of sheaf. □

Definition. Let X be a topological space and let $\mathcal{F}$ be a map from the set of open sets of X to the set of $\mathbb{K}$-algebras (or groups, rings,...). If $\mathcal{F}$ satisfies properties (i)-(iv) of a sheaf it is called a *presheaf.* As for sheaves, the notion of *stalk of a presheaf at a point* can be defined in the same way. The *sheaf associated to a presheaf* $\mathcal{F}$ is the sheaf $\mathcal{F}+$ constructed in Proposition 16.10.

Exercise 16.11. Prove that $\mathcal{F}^*$ is indeed a sheaf and that for each $x \in X$ the stalk of $\mathcal{F}^+$ at x coincides with the stalk of $\mathcal{F}$ at x.

Exercise 16.12. Prove that, given a morphism of sheafs $\varphi : \mathcal{F} \to \mathcal{F}'$ over a topological space X, then the assignments $U \mapsto \operatorname{Im} \varphi_U$ and $U \mapsto \operatorname{coker} \varphi_U$ are presheaves over X.

Definition. Let X be a topological space and let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a morphism of sheaves. Then the *image sheaf* $\mathcal{I}m\varphi$ and the *cokernel sheaf* of φ are defined respectively as the sheaves associated to the presheaves of Exercise 16.12. The morphism φ is said to be an *epimorphism of sheaves* if $\mathcal{I}m\varphi = \mathcal{F}'$.

We can now to prove that the morphism in Example 16.6 is actually an epimorphism. But instead of having to explicitly compute the image sheaf, we can find a more direct way by again looking at the stalks. The criterion is the following.

Proposition 16.13. *Let X be a topological space and let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a morphism of sheaves. Then φ is an epimorphism if and only if for any $x \in X$ the (well-defined) map $\varphi_x : \mathcal{F}_x \to \mathcal{F}'_X$ that sends the class of (U, s) to the class of $(U, \varphi_U(s))$ is surjective.*

Proof: It is an easy exercise to see that φ_x is well-defined for any x. If φ is an epimorphism, then by definition $\mathcal{I}m\varphi = \mathcal{F}'$. This means that, for any $x \in X$ and any germ $s'_x \in \mathcal{F}'_x$ represented by an equivalence class (U, s') the element $s' \in \mathcal{F}'(U)$ is also in $(\mathcal{I}m\varphi)(U)$. This implies that, if for each $y \in U$ the germ in y defined by (U, s') is denoted by s'_y, then around each $y \in U$ the germ s'_y is given locally by a section in the image of some φ_V. In particular, there exists an open set V containing x and a section $s \in \mathcal{F}(V)$ such that the germ s'_x is the class of $(V, \varphi_V(s))$. Therefore s'_x is the image by φ_x of the germ of $\mathcal{F}_x$ represented by (V, s). This proves that φ_x is surjective for any $x \in X$.

Reciprocally, assume that all the maps φ_x are surjective. Take any open set $U \subset X$ and any section $s' \in \mathcal{F}'(U)$. To see that s' is also in $(\mathcal{I}m\varphi)(U)$ we need to prove that, writing again s'_x for the germ at x given by the class of (U, s'), for each $x \in U$ this germ is given locally by a section in the image of some φ_V. But this is now an immediate consequence of the fact that φ_x is surjective. $\square$

This result implies immediately that for any embedded affine scheme structure the morphism $\mathcal{O}_{\mathbb{A}^n} \to i_*\mathcal{O}_X$ is an epimorphism, since the epimorphism $\mathbb{K}[X_1, \ldots, X_n] \to \mathbb{K}[X_1, \ldots, X_n]/I$ induces epimorphisms when localizing at any I_x for $x \notin V(I)$. Since an embedded projective scheme is locally an embedded affine scheme we also get epimorphisms $\mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_X$ in the projective case. We can therefore give the following general definition.

Definition. Let X, Y be scheme structures over affine or projective varieties X_{red} and Y_{red}. Assume $Y_{red} \subset Y_{red}$ and let i be the inclusion map. Then Y is said to be a *subscheme of the scheme* X if there is an epimorphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$. The kernel of this epimorphism is called the *sheaf of ideals of the subscheme Y inside the scheme S*.

17. Vector bundles

This chapter is included to motivate the definition of coherent sheaf in the next chapter (anyway it contains interesting material by itself). We start with some illustrative examples that will naturally yield to the definition of vector bundle if we want to "glue" affine equations.

Example 17.1. Let $X \subset \mathbb{P}^n$ be a hypersurface. Then we know that $I(X)$ is generated by some homogeneous polynomial $F \in \mathbb{K}[X_0, \dots, X_n]$ of degree say d. This is fine, except from the fact that F is not a function on $\mathbb{P}^n$. However if we consider the intersection of X with the affine space $D(X_i)$, it is now defined by a true function, namely the dehomogenization of F with respect of the variable X_i. Let us try to analyze why it is not possible to glue all these functions to get a function on $\mathbb{P}^n$. If we write the coordinates in $D(X_i)$ as $\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}$, then the function defining $X \cap D(X_i)$ is $\frac{F}{X_i^d}$. Therefore the reason why we cannot glue together the functions on $D(X_i)$ and $D(X_j)$ is that the second is obtained from the first one when multiplying by $\frac{X_i^d}{X_j^d}$.

If now $X \subset \mathbb{P}^n$ has arbitrary codimension r, one would have a priori the right to expect that X could be defined by exactly r equations. We know that this is not always the case (see for instance Example 1.10), and in fact this happens rarely (this is what we called complete intersection in Chapter 9). However it is true that good sets X (for instance smooth projective sets) enjoy that property locally around each point (this is implicitly stated in the proof of Theorem 11.1). The question now is: is it possible somehow to glue together those local functions in a similar way as in the previous example? Although the answer is negative in general, we present a positive case in order to give a clearer idea of what we mean.

Example 17.2. Let $X \subset \mathbb{P}^2$ be the subset consisting of the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. At each open set $U_i := D(X_i)$, $X \cap U_i$ consists of one point, so its affine ideal is generated by exactly two elements. More precisely, the ideals of $X \cap U_0$, $X \cap U_1$ and $X \cap U_2$ are respectively generated by $\frac{X_1}{X_0}, \frac{X_2}{X_0}$; $\frac{X_0}{X_1}, \frac{X_2}{X_1}$; and $\frac{X_0}{X_2}, \frac{X_1}{X_2}$. For any $\lambda \in \mathbb{K} \setminus \{0\}$ (the reader will understand soon why we do not take just $\lambda = 1$), it is possible to relate the two first sets of generators by the expression

$$\begin{pmatrix} \frac{X_1}{X_0} \\ \\ \frac{X_2}{X_0} \end{pmatrix} = \begin{pmatrix} (\frac{X_1}{X_0})^2 & 0 \\ \\ (1-\lambda)\frac{X_1X_2}{X_0^2} & \lambda\frac{X_1}{X_0} \end{pmatrix} \begin{pmatrix} \frac{X_0}{X_1} \\ \\ \frac{X_2}{X_1} \end{pmatrix}$$

and the matrix in the expression has entries regular in $U_0 \cap U_1$ and is invertible in that open set. Similarly, there is a relation

$$\begin{pmatrix} \frac{X_0}{X_1} \\ \\ \frac{X_2}{X_1} \end{pmatrix} = \begin{pmatrix} \lambda\frac{X_2}{X_1} & (1-\lambda)\frac{X_0X_2}{X_1^2} \\ \\ 0 & (\frac{X_2}{X_1})^2 \end{pmatrix} \begin{pmatrix} \frac{X_0}{X_2} \\ \\ \frac{X_1}{X_2} \end{pmatrix}$$

Putting together the last two relations we get

$$\begin{pmatrix} \frac{X_1}{X_0} \\ \\ \frac{X_2}{X_0} \end{pmatrix} = \begin{pmatrix} \lambda\frac{X_1X_2}{X_0^2} & (1-\lambda)\frac{X_2}{X_0} \\ \\ \lambda(1-\lambda)(\frac{X_2}{X_0})^2 & (\lambda^2-\lambda+1)\frac{X_2^2}{X_0X_1} \end{pmatrix} \begin{pmatrix} \frac{X_0}{X_2} \\ \\ \frac{X_1}{X_2} \end{pmatrix}$$

Thus if we want the matrix in this last relation to be invertible (and similar to the previous ones), we have to take λ such that $\lambda^2-\lambda+1=0$. This is possible since $\mathbb{K}$ is algebraically closed (and observe also that $\lambda(1-\lambda)=1$ and in particular $\lambda\neq 0,1$). We can therefore use those matrices to "glue together" the local equations of the points.

I want to remark that the reader who does not feel comfortable with imaginary numbers (as it is λ if $\mathbb{K}=\mathbb{C}$) can take different constants λ and λ' in the two first matrices, and for instance the choice $\lambda'=\frac{1}{1-\lambda}$ will work (I decided not to do so, because then the natural temptation is to take $\lambda=2,\lambda'=-1$, but this would not be honest in characteristic two).

Exercise 17.3. Repeat a similar trick for a set of four points in $\mathbb{P}^2$. Observe that the only interesting case is when three and only three of the points are on a line (since otherwise the ideal of the points is already generated by two homogeneous polynomials).

The following definition (if not already known by the reader) should look now very natural.

Definition. A *vector bundle of rank r* over a quasiprojective set X is a set $\mathbf{F}$ together with a map $p:\mathbf{F}\to X$ such that there exists a covering of X by a family of open sets $\{U_i\}_{i\in I}$ satisfying the following conditions:

(i) For each $i\in I$ there exists a bijection $\varphi_i : p^{-1}(U_i)\to U_i\times\mathbb{K}^r$ such that for each $x\in U_i$ and $v\in\mathbb{K}^r$ $p\varphi_i^{-1}(x,v)=x$.

(ii) For each $i,j\in I$ the map $\varphi_{ij}:=\varphi_j\circ\varphi_i^{-1}:(U_i\cap U_j)\times\mathbb{K}^r\to(U_i\cap U_j)\times\mathbb{K}^r$ is defined by $\varphi_{ij}(x,v)=(x,A_{ij}(x)(v))$, where A_{ij} is an $r\times r$ matrix whose entries are regular maps $U_i\cap U_j\to\mathbb{K}$.

If $r=1$, $\mathbf{F}$ is called a *line bundle.* The set $X\times\mathbb{K}^r$ is called the *trivial vector bundle of rank r* over X. We will usually write $\mathbf{F}_{|U}$ to denote $p^{-1}(U)$ for an open set $U\subset X$.

Notation. A vector bundle is clearly defined by an open covering $\{U_i\}_{i\in I}$ and the matrices A_{ij} on each $U_i\cap U_j$. Even if we will denote frequently a vector bundle with just a name $\mathbf{F}$, we will implicitly assume the map p, the open covering and the matrices to be also given.

And reciprocally, we will also assume a vector bundle to be defined if we already have the covering and the matrices.

The intuitive idea is that a vector bundle on X consists of assigning to each point of X a vector space of dimension r, and that all these vector spaces are glued in a regular way. In the definition we referred to $\mathbf{F}$ just as a set, but in fact the maps φ_i endow $\mathbf{F}$ with a structure of abstract algebraic variety (in the same way for instance as abstract differential manifolds are defined).

Example 17.4. Let us consider $\mathbf{U} \subset \mathbb{P}^n \times \mathbb{K}^{n+1}$ to be the subset of pairs (p, v) such that v is a vector in the vector line of $\mathbb{K}^{n+1}$ defining the point $p \in \mathbb{P}^n$. Then the first projection $\mathbf{U} \to \mathbb{P}^n$ endows $\mathbf{U}$ with the structure of a line bundle over $\mathbb{P}^n$ (called the *tautological line bundle over* $\mathbb{P}^n$). Indeed, for any $i = 0, \ldots, n$, we consider the open set $D(X_i)$. Writing any $p \in D(X_i)$ as $(a_0 : \ldots : a_{i-1} : 1 : a_{i+1} : \ldots : a_n)$, we get that a a vector $v \in \mathbb{K}^{n+1}$ such that $(p, v) \in \mathbf{U}$ can be written in a unique way as $v = (\lambda a_0, \ldots, \lambda a_{i-1}, \lambda, \lambda a_{i+1}, \ldots, \lambda a_n)$. We have therefore a bijection $\psi_i : D(X_i) \times \mathbb{K} \to p^{-1}(D(X_i))$ given by

$$\psi_i\big((b_0 : \ldots : b_n), \lambda\big) = \big((b_0 : \ldots : b_n), (\lambda \frac{b_0}{b_i}, \ldots, \lambda \frac{b_n}{b_i})\big)$$

The map $\psi_j^{-1} \circ \psi_i : D(X_i X_j) \times \mathbb{K} \to D(X_i X_j) \times \mathbb{K}$ is thus defined by

$$\psi_j^{-1} \circ \psi_i\big((b_0 : \ldots : b_n), \lambda\big) = \big((b_0 : \ldots : b_n), \lambda \frac{b_j}{b_i}\big).$$

Since $(b_0 : \ldots : b_n) \mapsto \frac{b_j}{b_i}$ is a regular function over $D(X_i X_j)$, the maps $\varphi_i := \psi_i^{-1}$ define a line bundle structure over $\mathbb{P}^n$.

Exercise 17.5. In a similar (and more general) way, consider the subset $\mathbf{U} \subset \mathbb{G}(k, n) \times \mathbb{K}^{n+1}$ consisting of pairs (Λ, v) such that v is a vector in the $(k+1)$-dimensional linear subspace of $\mathbb{K}^{n+1}$ defining Λ. Show that the second projection over $\mathbb{G}(k, n)$ defines on $\mathbf{U}$ a structure of vector bundle of rank $k+1$ (called the *tautological subbundle of the Grassmannian*).

Since a vector bundle consists of defining a vector space at each point of X, it is very natural to expect that we can define on vector bundles the same operations as in vector spaces.

Definition. We define the *dual of a vector bundle* $p : \mathbf{F} \to X$ to be the vector bundle $\mathbf{F}^*$ such that on each U_i (in the same open covering of X defining $\mathbf{F}$) we have $U_i \times (\mathbb{K}^r)^*$. The glueing $U_i \cap U_j \times (\mathbb{K}^r)^* \to U_i \cap U_j \times (\mathbb{K}^r)^*$ is defined by the dual of φ_{ij}, i.e. by $(x, v) \mapsto (x, A'_{ij}(x)(v))$, where A'_{ij} is the transpose of the inverse of A_{ij} (observe that since

$A_{ij}(x)$ is an isomorphism for any $x \in U_i \cap U_j$, the determinant of A_{ij} is nowhere zero function on $U_i \cap U_j$, so that the entries of A'_{ij} are also regular functions on $U_i \cap U_j$). In a similar way, it is possible to define the *k-th symmetric product* $S^k\mathbf{F}$ *of a vector bundle* $\mathbf{F}$ and the *k-th skew-symmetric product* $\bigwedge^k \mathbf{F}$ *of a vector bundle* $\mathbf{F}$.

Example 17.6. Consider for instance the dual $\mathbf{U}^*$ of the tautological line bundle over $\mathbb{P}^n$. An element of $\mathbf{U}^*$ can be regarded as the class of a pair $(p, L) \in \mathbb{P}^n \times (\mathbb{K}^{n+1})^*$, where two pairs (p, L) and (p, L') are identified if and only if $L - L'$ is identically zero on the vector line of $\mathbb{K}^{n+1}$ defining p. In other words, L can be interpreted as a linear form defined on the line defining p. We have now identifications $\psi'_i : D(X_i) \times \mathbb{K} \to \mathbf{U}^*_{|D(X_i)}$ assigning to each $\big((b_0 : \ldots : b_n), \lambda\big)$ the pair consisting of the same point $(b_0 : \ldots : b_n)$ and the linear form that assigns to the vector $(b_0, \ldots, b_n)$ the number λb_i. Therefore

$$\psi'^{-1}_j \circ \psi'_i\big((b_0 : \ldots : b_n), \lambda\big) = \big((b_0 : \ldots : b_n), \lambda \frac{b_i}{b_j}\big).$$

Example 17.7. Consider now the vector bundle $S^d\mathbf{U}^*$ over $\mathbb{P}^n$. Since the symmetric product of the linear forms is the vector space of homogeneous forms of degree d, we can identify an element of $S^d\mathbf{U}^*$ with a pair of a point $p \in \mathbb{P}^n$ and a homogeneous form F of degree d over the vector line of $\mathbb{K}^{n+1}$ defining p. In this case we have identifications on each $D(X_iX_j)$ given by

$$\big((b_0 : \ldots : b_n), \lambda\big) \mapsto \big((b_0 : \ldots : b_n), \lambda (\frac{b_i}{b_j})^d\big).$$

Remark 17.8. In the same vein, it is possible to define the *direct sum of two vector bundles*, the *tensor product of two vector bundles*, or the *vector bundle of homomorphisms of two vector bundles*. The only warning is that it is necessary to choose the same partition for the two initial bundles. Of course this is possible, because if $\mathbf{F}_{|U}$ is trivial for some vector bundle $\mathbf{F}$ on X and some open set U of X, then clearly $\mathbf{F}_{|V}$ is also trivial for any open set $V \subset U$. Having this in mind, if $\mathbf{F}$ and $\mathbf{F}'$ are two vector bundles over X of respective ranks r and r', with common open covering $\{U_i\}_{i \in I}$ and respective matrices A_{ij} and A'_{ij}, then we define the bundle $\mathbf{Hom}(\mathbf{F}, \mathbf{F}')$ locally by the corresponding $U_i \times Hom(\mathbb{K}^r, \mathbb{K}^{r'})$. If we identify a homomorphism from $\mathbb{K}^r$ to $\mathbb{K}^{r'}$ with a $r \times r'$-matrix M, the glueing homomorphisms $Hom(\mathbb{K}^r, \mathbb{K}^{r'}) \to Hom(\mathbb{K}^r, \mathbb{K}^{r'})$ on each $U_i \cap U_j$ are then given by $A \mapsto A'_{ij} A A_{ij}^{-1}$

Definition. A *regular section of a vector bundle* $p : \mathbf{F} \to X$ is a map $s : X \to \mathbf{F}$ such that $p \circ s = id_X$ and for each $i \in I$ the map $\varphi_i \circ s : U_i \to U_i \times \mathbb{K}^r$ is regular.

Example 17.9. The universal bundle of any Grassmannian does not have regular sections apart from the zero section. Indeed assume there is a regular section $s : \mathbb{G}(k,n) \to \mathbf{U}$. Composing with the inclusion in $\mathbb{G}(k,n) \times \mathbb{K}^{n+1}$ and projecting onto the second factor, we then obtain a regular map $\mathbb{G}(k,n) \to \mathbb{K}^{n+1}$. But each of its components must be a regular function on $\mathbb{G}(k,n)$, hence constant (after Exercise 7.14). Thus there exists $v \in \mathbb{K}^{n+1}$ such that $s(\Lambda) = (\Lambda, v)$. But v must be a vector in all the linear subspaces defining any $\Lambda \in \mathbb{G}(k,n)$, so that necessarily $v = (0, \dots, 0)$.

Example 17.10. Consider now the set of sections of the vector bundle $S^d\mathbf{U}^*$ over $\mathbb{P}^n$. If $s : \mathbb{P}^n \to S^d\mathbf{U}*$ is such a section , then at each $D(X_i)$ it is defined by $s(p) = (p, f_i(p))$, where $f_i : D(X_i) \to \mathbb{K}$ is a regular function, hence (after Theorem 14.5) a polynomial in $\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}$, or equivalently $f_i = \frac{F_i}{X_i^{a_i}}$, where each $F_i \in \mathbb{K}[X_0, \dots, X_n]$ $(i = 0, \dots, n)$ is a homogeneous polynomial of degree a_i (we can clearly take $a_i \geq d$). The compatibility conditions for $S^d\mathbf{U}^*$ we found in Example 17.7 say that for $i \neq j$ it should hold $f_j = f_i(\frac{X_i}{X_j})^d$, i.e. $F_j X_i^{a_i - d} = F_i X_j^{a_j - d}$. This means that (reducing denominators in f_i) we can take each a_i equal to d and then F_i is the same polynomial F for each $i = 0, \dots, n$. Therefore, the set of regular sections of $S^d\mathbf{U}^*$ is naturally identified with the set of homogeneous polynomials of degree d in $n+1$ variables.

Exercise 17.11. Show that the set of sections of the vector bundle $S^d\mathbf{U}^*$ over $\mathbb{G}(k,n)$ is also naturally identified with the set of homogeneous polynomials of degree d in $n+1$ variables (if you find a reasonable way of solving this exercise, please tell me).

Remark 17.12. The set of regular sections of a vector bundle $\mathbf{F}$ over X is a module over the ring $\mathcal{O}(X)$ of regular functions of X. Indeed let $s : X \to \mathbf{F}$ be a regular section and $f : X \to \mathbb{K}$ a regular map. Then fs is defined as the map $X \to \mathbf{F}$ which on each U_i is determined by $x \mapsto \varphi_i^{-1}(x, f(x)\psi_i(x))$, $\psi_i : U_i \to \mathbb{K}^r$ being the composition of $\varphi_i \circ s$ with the second projection. In other words, since $\mathbf{F}$ is locally (via φ_i) like $U_i \times \mathbb{K}^r$, then if s is locally defined by $x \mapsto (x, \psi_i(x))$, then fs is locally defined by $x \mapsto (x, f(x)\psi_i(x))$. It is an easy exercise to check that fs is globally defined as a regular section of $\mathbf{F}$. In a similar way, the sum of two regular sections s and s', locally defined by maps ψ_i and ψ'_i is defined by adding at each point x the vectors $\psi_i(x)$ and $\psi'_i(x)$. For instance the set of regular sections of a vector bundle over a projective variety is a vector space. Example 17.10 and Exercise 17.11 show that the vector space of the regular sections of $S^d\mathbf{U}^*$ is naturally isomorphic to the vector space of homogeneous polynomials of degree d in $n+1$ variables.

Exercise 17.13. Prove that the zero-locus of a regular section of a vector bundle over X is closed in X. Conclude that two sections of a vector bundle over a quasiprojective variety are equal if they coincide on an open set of X.

Example 17.14. Example 17.2 shows that if $X \subset \mathbb{P}^2$ is a set of three points in general position, then there exists a vector bundle on $\mathbb{P}^2$ having a regular section whose zero locus is exactly X. Exercise 17.3 proves the same when X is a set of four points such that exactly three of them lying on a line. If X is just one point, $I(X)$ is generated by two linear forms, so that X can be obtained as the zero locus of a regular section of $\mathbf{U}^* \oplus \mathbf{U}^*$. Similarly, if X is a set of two points $I(X)$ is then generated by a linear form and a quadratic form, so that X is the zero-locus of a regular section of $\mathbf{U}^* \oplus S^2\mathbf{U}^*$. In the same way, d points on a line are produced by a regular section of $\mathbf{U}^* \oplus S^d\mathbf{U}^*$, and four points in general position are obtained as the zero-locus of a section of $S^2\mathbf{U}^* \oplus S^2\mathbf{U}^*$. In general, it is true that any finite set of points in $\mathbb{P}^2$ is the zero-locus of some vector bundle of rank two over $\mathbb{P}^2$.

Exercise 17.15. Consider the smooth quadric $X = V(X_0X_1 + X_2X_3 + X_4^2) \subset \mathbb{P}^4$ and the line $L = V(X_0, X_2, X_4)$ contained in it. Prove that there exists a vector bundle $\mathbf{F}$ on Q with a section vanishing exactly at L. In fact it holds that there is a bijection among the set of sections of $\mathbf{F}$ (up to multiplication by a constant) and the set of lines contained in Q. If you can give a simple proof of it, I would be happy of seeing it.

Theorem 17.16. *Let X be an affine variety and $\mathbf{F}$ a vector bundle of rank r over X. Write P for the module of regular sections of $\mathbf{F}$.*

(i) For any $f \in \mathcal{O}(X)$, the set of sections of $\mathbf{F}_{|D_X(f)}$ is naturally isomorphic to P_f.

(ii) P is a finitely generated module over the ring $\mathcal{O}(X)$.

Proof: To prove (i), cover X with a finite number of open sets $D_X(f_i)$ on which $\mathbf{F}$ is trivial. If s is a regular section of $\mathbf{F}_{|D_X(f)}$, then each $s_{|D_X(ff_i)}$ is represented by r regular functions on $D_X(ff_i)$. Hence there exists an integer a_i such that $f^{a_i}s$ is a regular section on $D_X(f_i)$. Taking a to be the maximum of these a_i, we get on each $D_X(f_i)$ a regular section $f^a s$. Since they glue together by the same glueing matrices for s we thus obtain that $f^a s$ is in fact a regular section $s' \in P$. Therefore we can write $s = \frac{s'}{f^a}$, which proves (i).

For (ii) We still take a covering of X by open sets $D_X(f_i)$ on which $\mathbf{F}$ is trivial. For each i we define r regular sections $s'_{i1}, \ldots, s'_{ir} : D_X(f_i) \to \mathbf{F}_{D_X(f_i)}$ each s'_{ij} is given by the j-th coordinate vector, i.e. defined by $x \mapsto \varphi_i^{-1}(x, (0, \ldots, 0, \overset{j)}{1}, 0, \ldots, 0))$. By (i) we can find $a_i \in \mathbb{Z}$ such that $s_{ij} := f_i^{a_i} s'_{ij}$ is a regular section of $\mathbf{F}$. We claim that the sections s_{ij} generate the module of all the regular sections of $\mathbf{F}$.

Let $s : X \to \mathbf{F}$ be a regular section of $\mathbf{F}$. Restricted to each $D_X(f_i)$, it can be written in the form $s = \Sigma_j h_{ij} s_{ij}$, where the functions h_{ij} are regular on $D_X(f_i)$. Taking b_i to be the maximum exponent of f_i in the denominators of the h_{ij}'s, we write then

$$f_i^{b_i} s = \Sigma_j g_{ij} s_{ij}$$

with $g_{ij} \in \mathcal{O}(X)$ (both sides are equal on X since they coincide on the open set $D_X(f_i)$). On the other hand, since $\bigcup_i D_X(f_i^{b_i}) = X$, it follows that $1 = \Sigma_i k_i f_i^{b_i}$ for some $k_i \in \mathcal{O}(X)$. Therefore s can be written as $s = \Sigma_i k_i f_i^{b_i} s$. Substituting in the above displayed equation, we prove that the s_{ij}'s indeed generate the module of regular sections of $\mathbf{F}$. □

Notation. The module of regular sections of a vector bundle $\mathbf{F} \to X$ will be denoted by $\Gamma(X, \mathbf{F})$. From what we have seen, the assignation $U \mapsto \Gamma(U, \mathbf{F}_{|U})$ (for each open set $U \subset X$) defines a sheaf of $\mathcal{O}_X$-modules, which we will denote by $\mathcal{F}$ (the corresponding calligraphic letter), and it is called the *sheaf of sections of the vector bundle* $\mathbf{F}$. By abuse of notation we will often write $\Gamma(U, \mathcal{F})$ to indicate $\mathcal{F}(U)$. Some other standard notation is to call $\mathcal{O}_{\mathbb{P}^n}(d)$ for the sheaf of sections of the line bundle $S^d\mathbf{U}^*$ over $\mathbb{P}^n$.

Theorem 17.16(i) shows that the set of sections of the restriction of $\mathbf{F}$ to $D_X(f_i)$ is the localization of the module $\Gamma(X, \mathbf{F})$ at f_i, and this localization is a free $\mathcal{O}(X)_{f_i}$-module of rank r. This also shows that, for each $x \in X$, if $I_x \subset \mathcal{O}(X)$ is the maximal ideal of regular functions vanishing at x, then the vector space $p^{-1}(x)$ can be naturally identified with $\Gamma(X, \mathbf{F})/I_x\Gamma(X, \mathbf{F})$ (notice that this is a module over $\mathcal{O}(X)/I_x \cong \mathbb{K}$). On the other hand, for any $\mathcal{O}(X)$-module P it is possible to assign to each $x \in X$ the vector space P/I_xP, so a natural question is to ask when such an assignment corresponds to a vector bundle. Before giving an answer, we prove the following lemma.

Lemma 17.17. *Let P be a finitely generated $\mathcal{O}(X)$-module. Then for any $k \in \mathbb{Z}$ the set $Z_k := \{x \in X \mid \dim_{\mathbb{K}}(P/I_xP) \geq k\}$ is a Zariski closed set of X.*

Proof: Let x be a point outside Z_k and fix a set of generators $p_1, \ldots, p_s$ of P. The fact that $x \notin Z_k$ implies that, after changing order if necessary, the classes of $p_k, \ldots, p_s$ modulo I_xP linearly depend on the classes of $p_1, \ldots, p_{k-1}$. In other words, we can find relations

$$\begin{aligned} p_k &= \lambda_{k1}p_1 + \ldots + \lambda_{k,k-1}p_{k-1} + f_{k1}p_1 + \ldots + f_{ks}p_s \\ &\vdots \\ p_s &= \lambda_{s1}p_1 + \ldots + \lambda_{s,k-1}p_{k-1} + f_{s1}p_1 + \ldots + f_{ss}p_s \end{aligned}$$

with $\lambda_i \in \mathbb{K}$ and $f_{ij} \in I_x$ (i.e. they are regular functions such that $f_{ij}(x) = 0$). In a matricial way, the above relations yield

$$\begin{pmatrix} \lambda_{k1} + f_{k1} & \ldots & \lambda_{k,k-1} + f_{k,k-1} & -1 + f_{kk} & \ldots & f_{ks} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \lambda_{s1} + f_{s1} & \ldots & \lambda_{s,k-1} + f_{s,k-1} & f_{sk} & \ldots & -1 + f_{ss} \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_{k-1} \\ p_k \\ \vdots \\ p_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

But now if we write $f = \det\begin{pmatrix} -1+f_{kk} & \dots & f_{ks} \\ \vdots & \ddots & \vdots \\ f_{sk} & \dots & -1+f_{ss}\end{pmatrix}$, it follows that for any $y \in D_X(f)$, the classes of $p_k, \dots, p_s$ modulo I_yP still depend on the classes of $p_1, \dots, p_{k-1}$. Hence $x \in D_X(f) \subset X \setminus Z_k$, which proves that $X \setminus Z_k$ is open. □

Definition. Given a finitely generated $\mathcal{O}(X)$-module, the minimum $r \in \mathbb{Z}$ for which $Z_r \setminus Z_{r+1} \neq \emptyset$ is called the *rank of the module.*

Obviously, if P is the set of regular sections of some vector bundle over X of rank r, then P has rank r and $Z_{r+1} = \emptyset$. Surprisingly enough, this property characterizes the set of modules over $\mathcal{O}(X)$ that are obtained as regular sections of a vector bundle.

Definition. A *projective module* over a ring $\mathcal{O}$ is a module P satisfying the property that for any epimorphism $\varphi : M \to N$ of modules, any homomorphism $\psi : P \to N$ can be lifted to M, i.e. there exists a homomorphism $\phi : P \to M$ such that $\psi = \varphi \circ \phi$.

Theorem 17.18. *Let X be an affine variety, and let P be a finitely generated $\mathcal{O}(X)$-module of rank r. Then the following conditions are equivalent.*

(i) There exists a vector bundle $\mathbf{F}$ of rank r over X such that $\Gamma(X, \mathbf{F})$ is isomorphic to P.

(ii) P is a projective module.

(iii) There exists a finitely generated $\mathcal{O}(X)$-module P' such that $P \oplus P'$ is a free module.

(iv) For each $x \in X$, $\dim_{\mathbb{K}}(P/I_xP) = r$.

Proof: We will prove all the implications in a cyclic way.

(i)⇒(ii) Let $\mathbf{F}$ be a vector bundle over X, and want to prove that $P := \Gamma(X, \mathbf{F})$ is a projective $\mathcal{O}(X)$-module. So assume that we have an epimorphism $\varphi : M \to N$ of $\mathcal{O}(X)$-modules and a morphism $\psi : P \to N$. By Theorem 17.16(i) each localization P_{f_i} is a free $\mathcal{O}(X)_{f_i}$-module (of rank r). So localizing φ and ψ at each f_i (and denoting the corresponding morphism with the subindex i) we can define morphisms $\phi_i : P_{f_i} \to M_{f_i}$ such that $\psi_i = \varphi_i \circ \phi_i$. Considering the maximum exponent b_i of the denominators of the image by ϕ_i of a (finite) set of generators of P, we conclude that $f_i^{b_i}\phi_i$ defines a morphism from P to M. As in the proof of the previous theorem, we have a relation $1 = \Sigma_i k_i f_i^{b_i}$. This allows us to define $\phi := \Sigma_i k_i f_i^{b_i}\phi_i$, which is a morphism from P to M such that $\psi = \varphi \circ \phi$. This proves that P is a projective module.

(ii)⇒(iii) This is a standard result of commutative algebra. Let $p_1, \dots, p_r$ be a finite set of generators of P as an $\mathcal{O}(X)$-module. We can thus define an epimorphism $\varphi : \mathcal{O}(X)^{\oplus r} \to P$ by $\varphi(f_1, \dots, f_r) = f_1p_1 + \dots + f_rp_r$. If we consider the identity map of P as the map ψ

in the definition of projective module, we thus get that there exists a homomorphism $\phi : P \to \mathcal{O}(X)^{\oplus r}$ such that $\varphi \circ \phi = id_P$. This easily implies that $\mathcal{O}(X)^{\oplus r} \cong P \oplus \ker \varphi$.

(iii)⇒(iv) If $P \oplus P' \cong \mathcal{O}(X)^{\oplus s}$, it easily follows that $\dim_{\mathbb{K}}(P/I_xP) + \dim_{\mathbb{K}}(P'/I_xP') = s$ for any $x \in X$. But on the other hand if P and P' has respective ranks r and r', then $r + r' = s$ and $\dim_{\mathbb{K}}(P/I_xP) \geq r$, $\dim_{\mathbb{K}}(P'/I_xP') \geq r'$ for any $x \in X$. Thus the result follows immediately.

(iv)⇒(i) Let x be any point of X. Since $x \notin X_{r+1}$, we can repeat the proof of Lemma 17.17 and find $f_x \in \mathcal{O}(X)$ such that for any $y \in D_X(f_x)$ the classes modulo I_yP of r of the generators of P generate P/I_yP. But in the proof of Lemma 17.17 we have seen more, namely that those r elements, say $p_1, \ldots, p_r$ actually generate P_{f_x} as a module over $\mathcal{O}(X)_{f_x}$. On the other hand, if there were a linear relation $g_1p_1 + \ldots + g_rp_r = 0$ with $g_i \in \mathcal{O}(X)_{f_x}$ for all $i = 1, \ldots, r$ and with some $g_i \neq 0$, then the Nullstellensatz would imply that there exists $y \in D_X(f_x)$ such that $g_i(y) \neq 0$. But then the class of p_i modulo I_yP would depend on the other $r-1$ classes, implying that $\dim_{\mathbb{K}}(P/I_yP) \leq r-1$, which is absurd. Therefore P_{f_x} is a free $\mathcal{O}(X)_{f_x}$-module of rank r.

The geometric idea is now that we have an open covering of X on which the localization of P is free (hence representing a trivial vector bundle on each of the open sets), and we need to glue together all these pieces in order to obtain a vector bundle. We thus consider for each x the set $D_X(f_x) \times \mathbb{K}^r$, consider the disjoint union of all these sets and want to quotient by a relation. For $x \neq y$, we want to identify $D_X(f_xf_y) \times \mathbb{K}^r \subset D_X(f_x) \times \mathbb{K}^r$ with $D_X(f_xf_y) \times \mathbb{K}^r \subset D_X(f_y) \times \mathbb{K}^r$ via some map φ_{xy}. To this purpose assume that $p_1, \ldots, p_r \in P$ is the $\mathcal{O}(X)_{f_x}$-basis we chose for P_{f_x} and that $q_1, \ldots, q_r \in P$ is the $\mathcal{O}(X)_{f_x}$-basis we chose for P_{f_y}. Then they still form two $\mathcal{O}(X)_{f_xf_y}$-bases for $P_{f_xf_y}$. This means that there is a relation

$$\begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = A_{xy} \begin{pmatrix} q_1 \\ \vdots \\ q_r \end{pmatrix}$$

where A_{xy} is a matrix with entries in $\mathcal{O}(X)_{f_xf_y}$ and nonzero determinant. We thus define the wanted φ_{xy} by $(z, v) \mapsto (z, A_{xy}v)$. It is now a straightforward computation that this identification provides a vector bundle $\mathbf{F}$ over X whose set of regular sections is naturally isomorphic to P. □

18. Coherent sheaves

In Theorem 17.18 we have seen that there is a correspondence among sheaves of sections of vector bundles over an affine set X and finitely generated projective modules over $\mathcal{O}(X)$. A natural question is to ask what (not necessarily projective) finitely generated modules over $\mathcal{O}(X)$ correspond to. We will see that they correspond to a particular type of modules, which we will call coherent.

The hint for this correspondence is in the proof of Theorem 17.18. A section of a vector bundle $\mathbf{F}$ of rank r is defined locally by r regular functions on open sets,or equivalently by maps to $\mathbb{K}^r$. But for each x, the source $\mathbb{K}^r$ is naturally identified with P/I_xP, where $P = \Gamma(X, \mathbf{F})$. Hence a section on an open set U can be viewed as a map s from U to the disjoint union $\coprod_{x\in U} P/I_xP$ such that the image of each x lies in its corresponding P/I_xP. The regularity condition can be expressed by saying that in some small open neighborhood of x (on which the vector bundle is trivial) all the values of s glue together as a regular function. Since there is a basis of open sets of X of the form $D_X(g)$, and the set of regular functions is $\mathcal{O}(X)_g$, we can find some $D_X(g)$ on which $\mathbf{F}$ is trivial. We have therefore identifications $\Gamma(D_X(g), \mathbf{F}) \cong (\mathcal{O}(X)_g)^r$ and $(\mathcal{O}(X)_g)^r \cong P_g$ (P_g being the set of quotients $\frac{s}{g^l}$, with $s \in P$ and $l \in \mathbb{N}$), yielding a canonical identification $\Gamma(D_X(g), \mathbf{F}) \cong P_g$. All this yields the following generalization.

Proposition 18.1. *Let $X \subset \mathbb{A}^n$ be an affine variety and let M be a module over $\mathcal{O}(X)$. Then there exists a sheaf $\mathcal{F}_M$ of $\mathcal{O}_X$-modules defined by assigning to each open set $U \subset X$ the module $\mathcal{F}_M(U)$ consisting of the maps $f : U \to \coprod_{x\in U} M/I_xM$ such that for each $x \in U$ it holds $f(x) \in M/I_xM$ and there exists $g_x \in \mathcal{O}(X)$ with the property that, on $D_X(g_x)$, f is defined by some $m_x \in M_{g_x}$, i.e. $f(y)$ is the class modulo $I_yM_{g_x}$ of m_x for each $y \in V$. Moreover, for any basic open set $D_X(g)$, $\mathcal{F}_M(D_X(g))$ is naturally isomorphic to M_g. In particular, $\mathcal{F}_M(X)$ is naturally isomorphic to M.*

Proof: Proving that $\mathcal{F}_M$ is a sheaf of $\mathcal{O}_X$-modules is so tedious (although straightforward) that I leave it to the reader. About the last statement, it is clear that we have a homomorphism $M_g \to \mathcal{F}_M(D_X(g))$, associating to each $m \in M_g$ the map $f : X \to \coprod_{x\in X} M/I_xM$ that assigns to each x the class of m modulo I_xM (observe that M/I_xM is naturally isomorphic to the quotient of any M_g modulo I_xM_g). What we have to prove is that any element $f \in \mathcal{F}_M(D_x(g))$ can be globally defined by an element on M_g.

So fix $f \in \mathcal{F}_M(D(g)\cap X)$. For each $x \in X$, take a basic open neighborhood $D_X(g_x) \subset D_X(g)$ of x such that on it f is defined by $m_x \in M_{g_x}$. Since $D_X(g_x) = D_X(g_x^l)$ for any $l \in \mathbb{N}$, we can assume that we can write each m_x like $m_x = \frac{m'_x}{g_x}$, with $m'_x \in M$. As usual, it is possible to cover $D_X(g)$ with a finite number of open sets $D_X(g_x)$. Indeed the fact that $D_X(g) \subset \bigcup_x D_X(g_x)$ is equivalent to the intersection of X with the affine set defined by the polynomials g_x be contained in $V(g)$. And by the Hilbert's Nullstellensatz, this is in turn equivalent to the fact that a power of g is in the ideal generated by $I(X)$ and the polynomials g_x (and hence a finite number of them is needed). We have thus a relation $g^l = \Sigma_x h_x g_x$, with $h_x \in \mathcal{O}(X)$. It is then immediate to check that f can be represented globally by $\frac{\Sigma_x h_x m'_x}{g^l}$, which is an element of M_g. □

Definition. A *coherent sheaf over an affine variety* X will be a sheaf of the form $\mathcal{F}_M$ for some finitely generated $\mathcal{O}(X)$-module M. The *rank of a coherent sheaf* $\mathcal{F}_M$ is the rank of the module M.

We thus see that the study of sheaves over affine varieties is essentially the same as the study of modules, and that those corresponding to sections of a vector bundle are exactly the projective modules. Hence a good starting point for projective varieties could be to try to produce in a similar way sheaves from graded modules.

For instance, if $X \subset \mathbb{P}^n$ is a projective variety, we start with $S(X)$ as a module over itself. If we expect a behavior similar to the affine case, this should produce the sheaf $\mathcal{O}_X$. We observe that each $U_i := D_X(X_i)$ is an affine variety and that $\mathcal{O}_{U_i}$ is produced (via the previous construction) by the dehomogenization with respect to X_i of the elements of $S(X)$. This dehomogenization can be identified with the set $S(X)_{(X_i)}$ of the quotients $\frac{F}{X_i^d}$, where $d \in \mathbb{N}$ and F is a homogeneous element of $S(X)$ of degree d. But now there is no natural way of identifying the ground field $\mathbb{K}$ from the homogeneous ideal of any point. The key idea at this point is to observe that in the above definition of $\mathcal{F}_M$, since the map f is locally well-defined around each x, one can substitute M/I_xM by the localization M_{I_x} and the same results hold. And now we find a nice identification: the localization of $S(X)_{(X_i)}$ at the maximal ideal of a point $x \in D_X(X_i)$ is naturally isomorphic to the set of quotients $\frac{F}{G}$, where F and G are homogeneous elements of $S(X)$ of the same degree and G does not belong to the homogeneous prime ideal I_x of x. This yields to the following definition and construction.

Notation. Let M be a graded module over a graded ring S. If $F \in S$ is a homogeneous element of degree d, we denote with $M_{(F)}$ to the module consisting of quotients $\frac{m}{F^e}$, where $e \in \mathbb{N}$ and $m \in M$ is homogeneous of degree de. Similarly, if I is a prime homogeneous ideal of S, $M_{(I)}$ will denote the ring of the quotients $\frac{m}{F}$, where $m \in M$ and $F \in S$ are homogeneous of the same degree and $F \notin I$.

Proposition 18.2. *Let $X \subset \mathbb{P}^n$ be a projective variety and let M be a graded $S(X)$-module. Then there exists a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}_M$ consisting in assigning to each open set $U \subset X$ the set $\mathcal{F}_M(U)$ of those maps $U \to \coprod_{x \in U} M_{(I_x)}$ such that for each $x \in U$ it holds $f(x) \in M_{(I_x)}$ and there exist some $G_x \in \mathbb{K}[X_0, \dots, X_n]$ and $m \in M_{G_x}$ satisfying that, for* any $y \in D(G_x) \cap X$, $f(y) = m$. *Moreover, for any homogeneous* $G \in \mathbb{K}[X_0, \ldots, X_n]$. *The restriction of* $\mathcal{F}_M$ *to the affine variety* $D(G) \cap X$ *coincides with the sheaf constructed in Proposition 18.1 for the module* $M_{(G)}$.

Proof: This is again tedious and straightforward, so I leave again the proof to the reader. □

Definition. A *coherent sheaf over a projective variety* X is a sheaf of the form $\mathcal{F}_M$ for some finitely generated graded $S(X)$-module.

Exercise 18.3. Prove that, if $X \subset \mathbb{P}^n$ is a projective variety and $M = \mathcal{O}(X)^r$, then $\mathcal{F}_M$ is the sheaf of sections of the trivial bundle over X of rank r.

Example 18.4. Consider now the projective space $\mathbb{P}^n$ (thus with coordinate ring $S = \mathbb{K}[X_0, \ldots, X_n]$) and the S-module $M = \mathbb{K}[X_0, \ldots, X_n]/(X_0, \ldots, X_n)$ (which is obviously isomorphic to $\mathbb{K}$). For any $i = 0, \ldots, n$, the module $M_{(X_i)}$ is zero. Indeed the class of any $\lambda \in \mathbb{K}$ is the class of λX_i (as an element of M) divided by X_i (as an element of S). Since the class of X_i is zero in M, we get that $M_{(X_i)}$ is zero. Therefore the sheaf $\mathcal{F}_M$ is zero at any open set $D(X_i)$, so it is the constant sheaf zero.

The question now is how to reconstruct the homogeneous module from the sheaf. It cannot be done by taking global sections, since we have seen that some vector bundles do not have nonzero regular sections. On the other hand, Example 18.4 shows that two different modules can provide the same sheaf (obviously the zero module will yield the zero sheaf). We have in fact the following.

Proposition 18.5. *Let* $X \subset \mathbb{P}^n$ *be a projective variety and let* M *and* M' *be two finitely generated graded* $S(X)$*-modules. Then the following are equivalent:*

(i) M *and* M' *define isomorphic sheaves.*

(ii) For each $i = 0, \ldots, n$*, the* $S(X)_(X_i)$*-modules* $M_{(X_i)}$ *and* $M'_{(X_i)}$*are isomorphic.*

(iii) There exists $l_0 \in \mathbb{N}$ *such that* $\oplus_{l \geq l_0} M_l \cong \oplus_{l \geq l_0} M'_l$ *as* $S(X)$*-modules.*

Proof: Since (ii) means that the modules $\mathcal{F}_M$ and $\mathcal{F}_{M'}$ are isomorphic when restricted at any $D(X_i) \cap X$, it is clear that (i) and (ii) are equivalent.

On the other hand, if (iii) holds then for any $i = 0, \ldots, n$ and any $m' \in M_{(X_i)}$, we can clearly write $m' = \frac{m}{X_i^l}$ for some $l \geq l_0$. Since thus $M_l \cong M'_l$ we have that m' corresponds to an element of $M'_{(X_i)}$. And by symmetry any element of $M'_{(X_i)}$ has a counterpart in $M_{(X_i)}$. Therefore (ii) holds.

So we are left to prove that (ii) implies (iii). Let $m_1, \ldots, m_r$ be a system of generators of M. Since each $\frac{m_j}{1}$, as an element of $M_{(X_i)}$, corresponds to an element of $M'_{(X_i)}$ via an isomorphism ϕ_i, there exists some $l_{ij} \in \mathbb{N}$ such that $X_i^{l_{ij}}\phi_i(m_j) \in M'_i$. Since $m_1, \ldots, m_r$ generate M, it follows that there exists some $l_0 \in \mathbb{N}$ such that $\oplus_{l \geq l_0} M_l$ is generated by the elements $X_i^{l_{ij}} m_j$. It is thus easy to show that there is a well-defined homomorphism $\oplus_{l\geq l_0} M_l \to \oplus_{l\geq l_0} M'_l$ sending each $X_i^{l_{ij}} m_j$ to $X_i^{l_{ij}}\phi_i(m_j) \in M'_i$. In a similar way one can construct a homomorphism $\oplus_{l\geq l_0} M'_l \to \oplus_{l\geq l_0} M_l$ (maybe increasing the value of l_0) that is inverse of the other one. This completes the proof. $\square$

If we just want to reconstruct the graded ring of a projective variety from its sheaves, we can find an idea at Example 17.10, since $\mathbb{K}[X_0, \ldots, X_n]$, which is the coordinate ring of $\mathbb{P}^n$, can be obtained as $\bigoplus_d \Gamma(\mathbb{P}^n, S^d\mathbf{U}^*)$. Thus we could wonder whether for any projective variety $X \subset \mathbb{P}^n$ its graded ring is isomorphic to $\bigoplus_d \Gamma(X, S^d\mathbf{U}^*_{|X})$. The following examples will give an idea of how close this is to be true.

Example 18.6. Let $X \subset \mathbb{P}^3$ be the rational quartic $X = \{(t_0^4 : t_0^3 t_1 : t_0 t_1^3 : t_1^4) \in \mathbb{P}^3 \mid (t_0 : t_1) \in \mathbb{P}^1\}$. It is easy to see (using the natural substitution map $\mathbb{K}[X_0, X_1, X_2, X_3] \to \mathbb{K}[T_0, T_1]$) that $S(X)_l \cong \mathbb{K}[T_0, T_1]_{4l}$ for $l \geq 2$, while $S(X)_1$ is (freely) generated by the classes of X_0, X_1, X_2, X_3, so that it is isomorphic to the subspace of $\mathbb{K}[T_0, T_1]$ generated by $T_0^4, T_0^3T_1, T_0T_1^3, T_1^4$. In other words, $T_0^2T_1^2$ is missing. Let us see that however this missing part can be obtained as a regular section of $\mathbf{U}^*_{|X}$. First of all, observe that X is covered by the two open sets $D(X_0)$ and $D(X_3)$. On those open sets, the expression $T_0^2T_1^2$ can be obtained respectively as $\frac{X_1^2}{X_0} = X_0\frac{X_1^2}{X_0^2}$ and $\frac{X_2^2}{X_3} = X_3\frac{X_2^2}{X_3^2}$. But if we glue the functions $\frac{X_1^2}{X_0^2}$ and $\frac{X_2^2}{X_3^2}$ by multiplying by $\frac{X_3}{X_0}$ we just get a regular section of $\mathbf{U}^*_{|X}$. What actually happens in this example (the reader is not expected to completely understand at this point why) is that X is isomorphic to $\mathbb{P}^1$, and $\mathbf{U}^*_{|X}$ over X corresponds to $S^4\mathbf{U}$ over $\mathbb{P}^1$, so that eventually $\bigoplus_d \Gamma(X, S^d\mathbf{U}^*_{|X})$ yields $\bigoplus_d \mathbb{K}[T_0, T_1]_{4d}$. So somehow $\bigoplus_d \Gamma(X, S^d\mathbf{U}^*_{|X})$ "completes" some "missing parts" of $S(X)$.

Example 18.7. In view of the above example, one can also think that something similar happens to the cubic $X = \{(t_0^3 : t_0t_1^2 : t_1^3) \in \mathbb{P}^2 \mid (t_0 : t_1) \in \mathbb{P}^1\}$. Now the missing part of $S(X)_1$, which is isomorphic to a subspace of $\mathbb{K}[T_0, T_1]_3$ is $T_0^2T_1$. Now X is covered by $D(X_0)$ and $D(X_2)$, and on $D(X_2)$ the expression $T_0^2T_1$ can be written as $\frac{X_1^2}{X_2} = X_2\frac{X_1^2}{X_2}$. However, on $D(X_0)$ it is not difficult to see that this expression cannot be obtained in such a way. What happens now is that X is not isomorphic to $\mathbb{P}^1$ (see Example 7.3), so that we cannot identify $\mathbf{U}^*_{|X}$ over X with $S^3\mathbf{U}$ over $\mathbb{P}^1$. And in fact it eventually happens that $S(X)$ is isomorphic to $\bigoplus_d \Gamma(X, S^d\mathbf{U}^*_{|X})$.

Exercise 18.8. Decide whether the curve $X = \{(t_0^4 : t_0^2t_1^2 : t_0t_1^3 : t_1^4) \in \mathbb{P}^3 \mid (t_0 : t_1) \in \mathbb{P}^1\}$ behaves like in Example 18.6 or like in Example 18.7 with respect to the regular sections of $\mathbf{U}^*_{|X}$ (of course you are not expected to give any proof, since we did not do in the previous examples).

The word "coherent" for a sheaf comes from the fact that the global behavior of a coherent sheaf can be derived from the local behavior at its points, i.e. by its stalks. To see this, we need first to find out what the stalk of a coherent sheaf is in terms of the module defining it.

Lemma 18.9. *Let X be an affine variety and let $\mathcal{F}_M$ be the coherent sheaf on X defined by the $\mathcal{O}(X)$-module M. If x is a point of X and $I \subset \mathcal{O}(X)$ is the maximal ideal corresponding to x, then that the stalk of $\mathcal{F}_M$ at x is naturally isomorphic to the localization M_I.*

Proof: It is a straightforward generalization of the proof of Proposition 16.7. □

Lemma 18.10. *Let X be a projective variety and let $\mathcal{F}_M$ be the coherent sheaf on X defined by the graded $S(X)$-module M. If x is a point of X and $I \subset S(X)$ is the prime ideal corresponding to x, then the stalk of $\mathcal{F}_M$ at x is naturally isomorphic to the localization $M_{(I)}$ consisting of quotients $\frac{s}{F}$, where $s \in M$ and $F \notin I$ are homogeneous of the same degree.*

Proof: It is the generalization of Proposition 16.9. □

The following result gives an idea of the "coherence" of coherent sheaves with respect to their stalks.

Theorem 18.11. *Let $\mathcal{F}$ be a coherent sheaf over a quasiprojective variety X. Then $\mathcal{F}$ is the sheaf of sections of a vector bundle $\mathbf{F}$ over X of rank r if and only if each stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X_x}$-module of rank r.*

Proof: Clearly it is enough to show the result when X is affine. One implication is obvious, since given a vector bundle $\mathbf{F}$on X, then for any $x \in X$ we can take a neighborhood of the type $D_X(f)$ such that the restriction of $\mathbf{F}$ to it is trivial. Therefore, if $\mathcal{F}$ is the sheaf of sections of $\mathbf{F}$, then the localization of $M := \mathcal{F}(X)$ in f is free of rank r. Hence the localization in the ideal of x in $\mathcal{O}(X)$ (i.e. the stalk of $\mathcal{F}$ at x) is also free of rank r.

Reciprocally, assume that we have a coherent sheaf $\mathcal{F}$ such that for each $x \in X$ the localization of $M := \mathcal{F}(X)$ at the ideal I_x of x is free of rank r. Since M/I_xM is naturally isomorphic to the quotient by the maximal ideal of $O(X)_{I_x}$ of the localization M_{I_x}, it immediately follows that $\dim_{\mathbb{K}}(M/I_xM) = r$ for each $x \in X$. Theorem 17.18(iv) implies thus that $\mathcal{F}$ is the sheaf of sections of a vector bundle over X. □

Definition. A coherent sheaf as in Theorem 18.11 is called a *locally free sheaf*.

Since Lemma 18.10 forces us to work with localizations in primes, we can try to see what happens when localizing at arbitrary prime ideals. This is what we do in the following results.

Proposition 18.12. *Let X be an affine variety and let $\mathcal{F}$ be a coherent sheaf on X defined by a module M. Given $I \subset \mathcal{O}(X)$ a prime ideal, let $K(I)$ be the quotient field of $\mathcal{O}(X)/I$ (or equivalently the quotient of $\mathcal{O}(X)_I$ modulo its maximal ideal). Then the quotient of M_I modulo the maximal ideal of $\mathcal{O}(X)_I$ is a vector space over $K(I)$ whose dimension coincides with the rank of the restriction of $\mathcal{F}$ to $V(I)$. Therefore this dimension is the dimension of M/I_xM when x varies in an open set of $V(I)$.*

Proof: Let $m_1, \ldots, m_s$ be a set of generators of M as an $\mathcal{O}(X)$-module. The proof of Lemma 17.17 shows that if for some point $x \in V(I)$ we have that M/I_xM is generated, as a $\mathbb{K}$-vector space, by the classes of r elements among $m_1, \ldots, m_s$, say $m_1, \ldots, m_r$, then there exists some $f \notin I_x$ such that $m_{r+1}, \ldots, m_s$ are a linear combination of $m_1, \ldots, m_r$ with coefficients in $\mathcal{O}(X)_f$. Since therefore $f \notin I$, it follows that the classes of $m_1, \ldots, m_r$ generate M_I/IM_I as a $K(I)$-vector space. This shows that $\dim_{K(I)}(M_I/IM_I)$ is bounded by the rank of $\mathcal{F}_{|V(I)}$.

But reciprocally, if the class of say m_r modulo IM_I depends linearly on the classes of $m_1, \ldots, m_{r-1}$, this means that there exists some $f \notin I$ such that fm_r depends linearly on $m_1, \ldots, m_{r-1}$ modulo IM. By the affine Nullstellensatz, we can find $x \in V(I)$ such that $f(x) \neq 0$. Therefore m_r will depend linearly on $m_1, \ldots, m_{r-1}$ modulo I_xM. This shows now that the rank of $\mathcal{F}_{|V(I)}$ is at most $\dim_{K(I)}(M_I/IM_I)$, completing the proof. □

Proposition 18.13. *Let X be an projective variety and let $\mathcal{F}$ be a coherent sheaf on X defined by a graded module M. Given $I \subset S(X)$ a homogeneous prime ideal, let $K(I)$ be the set of quotients of the form $\frac{F}{G}$, with $F, G \in \mathcal{O}(X)/I$ homogeneous of the same degree (equivalently $K(I)$ is the quotient of $\mathcal{O}(X)_{(I)}$ modulo its maximal ideal). Then the quotient of $M_{(I)}$ modulo the maximal ideal of $\mathcal{O}(X)_I$ is a vector space over $K(I)$ whose dimension coincides with the rank of the restriction of $\mathcal{F}$ to $V(I)$.*

Proof: Again it can be proved easily by either imitating the proof of Proposition 18.12 or by using that result. □

19. Schemes

We are essentially ready to define the notion of scheme. Recall that we wanted to allow some extra structure to quasiprojective sets (in the sense of Example 1.21, in which we had a point together with some infinitesimal information). It seems clear a priori that an affine scheme should consist of an arbitrary ideal $I \subset \mathbb{K}[X_1, \ldots, X_n]$ and that the underlying affine set should be the set of maximal ideals of $\mathbb{K}[X_1, \ldots, X_n]/I$. Similarly, to have a projective scheme we would need a homogeneous ideal $I \subset \mathbb{K}[X_0, \ldots, X_n]$ and then the points would correspond to some particular prime ideals of $\mathbb{K}[X_0, \ldots, X_n]/I$. However, Propositions 18.12 and 18.13 seem to suggest that if we consider all the prime ideals we get some extra information. This is not actually a serious reason to include the set of primes in the definition (because the mentioned extra information can be obtained a posteriori from the maximal ideals). The true reason is that sometimes one needs to work with rings with "few" maximal ideals. The following example will hopefully explain this idea.

Example 19.1. Let us interpret Example 1.23 (see also Example 8.12) in a different (but more natural) context. The generators $X_1X_2, X_1X_3, X_2X_3 - tX_0X_2, X_3^2 - tX_0X_3$ of the ideal I_t can be regarded now as polynomials in $\mathbb{K}[X_0, X_1, X_2, X_3, t]$ that are homogeneous in X_0, X_1, X_2, X_3, hence defining a quasiprojective set $Y \subset \mathbb{P}^3 \times \mathbb{A}^1$. The second projection defines a regular map $f : Y \to \mathbb{A}^1$ and its fiber ideal for any $t \in \mathbb{A}^1$ is precisely I_t. As we have seen, the ideal I_0 has an embedded component, while for a general t (in fact if $t \neq 0$) I_t is the disjoint union of two lines. Following the idea of Proposition 18.12, this general behavior should be obtained when localizing $\mathbb{K}[t]$ at the zero ideal, i.e. when we consider $\mathbb{K}(t)$, the quotient field of $\mathbb{K}[t]$. Regard then the above equations as homogeneous equations in the polynomial ring $\mathbb{K}(t)[X_0, X_1, X_2, X_3]$ with coefficients in the field $\mathbb{K}(t)$. In this case the equality $(X_1X_2, X_1X_3, X_2X_3 - tX_0X_2, X_3^2 - tX_0X_3) = (X_2, X_3) \cap (X_1, X_3 - tX_0)$ still holds (because t is just a non-zero element of the field $\mathbb{K}(t)$), and then the ideal represents the disjoint union of the lines $V(X_2, X_3)$ and $V(X_1, X_3 - tX_0)$.

But if we want to study the infinitesimal behavior of the fiber of f at 0, we could have localize $\mathbb{K}[t]$ at the prime ideal (t), or (see Remark 16.8) to consider its inclusion in $\mathbb{K}[[t]]$. In both cases we get a ring with just two prime ideals: the maximal ideal (t) (the quotient of which gives us two meeting lines plus one embedded point in $\mathbb{P}^3_{\mathbb{K}}$) and the non-maximal ideal (0), which gives us two skew lines in $\mathbb{P}^3_{\mathbb{K}(t)}$ (or in $\mathbb{P}^3_{\mathbb{K}((t))}$ if we considered $\mathbb{K}[[t]]$). The geometric interpretation is that we have a formal infinitesimal deformation of the two meeting lines plus the embedded point, but this deformation can be viewed only when considering a non-maximal prime ideal. This kind of "virtual" deformations constitute the main reason to consider all the prime ideals when defining a scheme.

Definition. The *spectrum of a ring R* is the set $X = \operatorname{Spec}(R)$ consisting of all the prime ideals of R, endowed with the topology (called the *Zariski topology*) in which the closed sets are the sets of the form $V(I) := \{P \in \mathrm{Spec}(R) \mid P \supset I\}$ for some ideal $I \subset R$. An *affine scheme* is the spectrum X of a ring R together with a sheaf of rings $\mathcal{O}_X$ in which $\mathcal{O}_X(U)$ consists of the set of maps $s : U \to \coprod_{P \in U} R_P$ such that for each $P \in U$ $s(P) \in R_P$ and there exists a neighborhood V of P and elements $f, g \in R$ satisfying that for each $Q \in V$ it holds $s(Q) = \frac{f}{g}$ (in particular $g \notin Q$).

Observe that the definition of the Zariski topology and the structure sheaf of an affine scheme is completely analogous to the corresponding ones for affine sets. We thus have a series of similar results.

Lemma 19.2. *The following properties of an affine scheme $X = \mathrm{Spec}(R)$ hold:*

(i) The sets $D(f) := X \setminus V(f)$ with $f \in R$ form a basis of the Zariski topology on X.

(ii) The closure of the set $\{P\}$ is $V(P)$. In particular, $\{P\}$ is closed if and only if P is a maximal ideal.

(iii) If for any subset $Z \subset X$ we write $I(X) = \{f \in R \mid f \in P$ for each $P \in Z\}$ (i.e. $I(X) = \bigcup_{P \in X} P$), then the analogous properties to those of Proposition 1.1 are satisfied.

(iv) For any ideal $I \subset R$, it holds $IV(I) = \sqrt{I}$.

(v) For each $P \in X$, the stalk of $\mathcal{O}_X$ at P is naturally isomorphic to R_P.

Proof: We leave almost everything as en exercise, and prove only part (iv), since maybe the reader is scared thinking of having to prove some analog to the Nullstellensatz. By definition, $IV(I) = \bigcup_{P \in V(I)} P = \bigcap_{P \supset I} P$, and this is $\sqrt{I}$ by Exercise 0.1(vii). $\square$

Definition. A *scheme* is a topological space X endowed with a sheaf of rings $\mathcal{O}_X$ (called the *structure sheaf of the scheme*) such that X is covered by open sets U satisfying that U together with $\mathcal{O}_{X|_U}$ is isomorphic to an affine scheme.

Definition. If S is a homogenous ring, we will call the *projective spectrum* of S to the set $\mathrm{Proj}(S)$ of all the homogeneous prime ideals not containing all the homogenous elements of positive degree, endowed with the *Zariski topology* in which the closed sets have the form $V(I) := \{P \in \mathrm{Proj}(S) \mid P \supset I\}$. And a *projective scheme* will be a projective spectrum $X = \mathrm{Proj}(S)$ together with the *structure sheaf* $\mathcal{O}_X$ assigning to each open set U the ring $\mathcal{O}_X(U)$ of maps $s : U \to \coprod_{P \in U} R_{(P)}$ such that for each $P \in U$ $s(P) \in R_{(P)}$ and there exist an open neighborhood V of P and homogeneous elements $F, G, \in S$ such that for each $Q \in V$ it holds $G \notin Q$ and $s(Q) = \frac{F}{G}$.

Proposition 19.3. *With the above definition, $X = \mathrm{Proj}(S)$ becomes a scheme and, more precisely, for any homogeneous $F \in S$ it holds that $D(F)$ is an affine scheme with* *structure sheaf equal to* $\mathcal{O}_{X|D(F)}$. *In particular, for each* $P \in X$, *the stalk* $\mathcal{O}_X, P$ *is naturally isomorphic to* $S_{(P)}$.

Proof: It is just a standard straightforward computation. □

Remark 19.4. It is clear that for affine schemes and projective schemes we can extend our definitions (and properties) of coherent sheaves over affine or projective varieties. Even if the whole theory below can be done for sufficiently general rings, I would prefer not to reach such a deep degree of abstraction, so that we will restrict ourselves to rings that are finitely generated $\mathbb{K}$-algebras, i.e. the quotient of a polynomial ring by a arbitrary ideals.

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